Critical branching processes with immigration: scaling limits of local extinction sets

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Branching and Persistence
Conference at University of Angers

Engineering and Physical Sciences Research Council

April 9, 2025

EP/W006227/1 & EP/V009478/1

Plan of the talk

1. Setting and results

2. Proofs: structure and ideas

Bienaymé-Galton-Watson process with immigration (BGWI)

Offspring μ ($\sim (\xi_i^{(n)})_{i,n\in\mathbb{N}}$ IID) & **Immigration** ν ($\sim (\eta_n)_{n\geq 1}$ IID) laws on $\mathbb{N} := \{0, 1, \ldots\}$, satisfying $(\xi_i^{(n)})_{i,n\in\mathbb{N}} \perp\!\!\!\perp (\eta_n)_{n\geq 1}$:

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$$E \coloneqq \{m \in \mathbb{N} : Z_1(m) = 0\} \quad \& \quad L_1(n) \coloneqq |E \cap \{0,\dots n\}| \xrightarrow{\text{a.s.}} \infty \quad \text{as } n \to \infty.$$

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Main Question. Is there a scaling limit of L_1? More generally, is there joint scaling limit

$$\left(\frac{1}{b_n} Z_1(\lfloor n\cdot\rfloor), \frac{1}{c_n} L_1(\lfloor n\cdot\rfloor)\right) \quad \text{for some scaling sequences } b_n, c_n \to \infty?$$

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The formula $Z_1(0) = z \in \mathbb{N} \quad \& \quad Z_1(n+1) = \sum_{i=1}^{Z_1(n)} \xi_i^{(n)} + \eta_{n+1}, \; n \in \mathbb{N}$, implies

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Heuristic: if X_1 and Y_1 have scaling limits, Z_1 will be in a domain of attraction (DoA) of a self-similar continuous-state branching process with immigration (CBI).

Caution: scale X_1 and Y_1 in a **balanced** way so neither reproduction nor immigration dominate.

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Assumption (SL) The generating functions

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for $c, d > 0$ and $\alpha \in (0,1]$ and slowly varying functions $\ell, \hslash : (0,1) \to (0,\infty)$, s.t. $\frac{\ell(s)}{\hslash(s)} \xrightarrow{s\to 0} 1$

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Theorem ([CPGUB13]: $X_1 \in$RW$(\tilde{\mu})$, $Y_1 \in$RW(ν), $Z_1(n) = X_1(\sum_{i=1}^{n-1} Z_1(i)) + Y_1(n)$)

- $\left(X_1(\lfloor nb_n \cdot \rfloor), Y_1(\lfloor n\cdot \rfloor)\right)/b_n \xrightarrow{d} (X, Y)$, *where* $b_n^\alpha \sim n\ell(1/b_n) \sim n\hbar(1/b_n)$ *as* $n \to \infty$, α*-stable subordinator* Y *and* $(1+\alpha)$*-stable spectrally positive Lévy process* X*;*

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- $Z_1(\lfloor n\cdot\rfloor)/b_n \xrightarrow{d} Z$, *where* $Z = X(\int_0^\cdot Z(s)\,ds) + Y$ *is a self-similar CBI of index* α.

Joint scaling limit for BGWI Z_1 and counting local time L_1 under (SL)

Assumption (SL) The generating functions for reproduction μ and immigration ν:

$$f(s) = s + c(1-s)^{1+\alpha}\ell(1-s) \quad \& \quad g(s) = 1 - d(1-s)^{\alpha}\ell(1-s),$$

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Theorem (Main Limit Theorem in [MPUB25])

Let Assumption (SL) hold with $\delta = \frac{d}{\alpha c} \in (0,1)$. Then, for any $\kappa > 0$, the sequence (c_n),

$$c_n := \kappa n \mathbb{P}(Z_1(n) = 0), \quad n \in \mathbb{N},$$

is regularly varying of index $1-\delta$. There exists κ so that weak convergence (in Skorohod top.)

$$\left(\frac{1}{b_n} Z_1(\lfloor n\cdot \rfloor), \frac{1}{c_n} L_1(\lfloor n\cdot \rfloor)\right) \xrightarrow{d} (Z, L) \text{ as } n \to \infty$$

holds, where $b_n^\alpha \sim n\ell(1/b_n)$, $Z = X(\int_0^\cdot Z(s)\,ds) + Y$ α-self-similar CBI (α-stable subord. Y, $(1+\alpha)$-stable spectrally positive Lévy process X) and L Markov local time of Z at 0.

Remarks on the Main Limit Theorem

Under (SL) and $\delta = \frac{d}{\alpha c} \in (0,1)$, $\exists \kappa > 0$ s.t. regularly varying $c_n := \kappa n \mathbb{P}(Z_1(n) = 0)$ satisfies

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- Both (c_n) and Markov local time L defined up to a multiplicative constant.

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- $\delta \in (0,1)$ is necessary: if $\delta \geq 1$, Z is not point recurrent at 0 [FUB14]. For example, if $\alpha = 1$, then $Z \sim \frac{2}{c} \cdot \mathrm{BESQ}(2\delta)$, making 0 is polar.

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- **Weak limit** (1) **covers all** critical finite-variance offspring laws μ. Reproduction μ has finite variance $\iff$ $f''(1) < \infty$ $\iff$ $\alpha = 1$ & $\exists \lim_{s \downarrow 0} \ell(s) < \infty$. Then, by balance condition $\frac{\ell(s)}{\not{k}(s)} \to 1$ as $s \to 0$ in Assumption (SL), we have finite-mean immigration ν.

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- $\diamond$ Both (c_n) and Markov local time L defined up to a multiplicative constant.
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- $\diamond$ **Weak limit** (1) **covers all** critical offspring and immigration laws μ and ν in SDoA$(1+\alpha)$ and SDoA(α), respectively, for $\alpha \in (0,1)$.
- $\diamond$ **Weak limit** (1) **covers all** critical finite-variance offspring laws μ. Reproduction μ has finite variance $\iff$ $f''(1) < \infty$ $\iff$ $\alpha = 1$ & $\exists \lim_{s \downarrow 0} \ell(s) < \infty$. Then, by balance condition $\frac{\ell(s)}{\ell\!\!\!/(s)} \to 1$ as $s \to 0$ in Assumption (SL), we have finite-mean immigration ν.
- $\diamond$ If $\alpha = 1$ but $\boldsymbol{f''(1) = \infty}$, offspring law μ has **infinite variance** but scaling limit (1) equals $\frac{2}{c} \cdot \mathrm{BESQ}(2\delta)$ and its Markovian local time at 0.
- $\diamond$ Substituting (μ, ν) with (μ, ν_n), for immigration $\nu_n := (1 - p_n)\delta_0 + p_n \frac{1}{1-\nu(0)} \nu|_{\mathbb{N} \setminus \{0\}}$ with $p_n \uparrow 1$, produces $\frac{1}{b_n} Z_1^{(n)}(\lfloor n \cdot \rfloor) \xrightarrow{d} Z$ as in (1), but $\frac{1}{c'_n} L_1^{(n)}(\lfloor n \cdot \rfloor) \not\to L$ for any scaling (c'_n).

Septuple Limit Theorem

- $Z_1 = X_1 \circ C_1 + Y_1$, where $C_1 = \sum_{0\le j<\cdot} Z_1(j)$ and RW($\tilde{\mu}$)$\ni X_1 \perp\!\!\!\perp Y_1 \in$RW($\nu$).
- Let $L_1(X_1)$ and $L_1(Z_1)$ be the counting local times at 0 of X_1 and Z_1.

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- Let $L_1(X_1)$ and $L_1(Z_1)$ be the counting local times at 0 of X_1 and Z_1.
- $Z = X \circ C + Y$, where $C = \int_0^\cdot Z(s)\,ds$ and $X \perp\!\!\!\perp Y$ stable Lévy processes with Laplace exponents $\Psi(\lambda) = c\lambda^{1+\alpha}$ and $\Phi(\lambda) = d\lambda^\alpha$.
- Let $L(X)$ and $L(Z)$ be (Markovian) local times at 0 of X and Z.

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- Let $L_1(X_1)$ and $L_1(Z_1)$ be the counting local times at 0 of X_1 and Z_1.
- $Z = X \circ C + Y$, where $C = \int_0^\cdot Z(s)\,ds$ and $X \perp\!\!\!\perp Y$ stable Lévy processes with Laplace exponents $\Psi(\lambda) = c\lambda^{1+\alpha}$ and $\Phi(\lambda) = d\lambda^{\alpha}$.
- Let $L(X)$ and $L(Z)$ be (Markovian) local times at 0 of X and Z.

Theorem (Septuple limit theorem in [MPUB25])

Under (SL) with $\delta = \frac{d}{\alpha c} \in (0,1)$ and let (b_n), (c_n) be regularly varying scaling sequences as above. Then there exists a regularly varying sequence (a_n) of index $1 + 1/\alpha$, such that

$$\left(\frac{X_1(\lfloor nb_n \cdot \rfloor)}{b_n}, \frac{L_1(X_1)(\lfloor nb_n \cdot \rfloor)}{a_{nb_n}}, \frac{Y_1(\lfloor n \cdot \rfloor)}{b_n}, \frac{C_1(\lfloor n \cdot \rfloor)}{nb_n}, \frac{X_1 \circ C_1(\lfloor n \cdot \rfloor)}{b_n}, \frac{Z_1(\lfloor n \cdot \rfloor)}{b_n}, \frac{L_1(Z_1)(\lfloor n \cdot \rfloor)}{c_n}\right)$$

$$\xrightarrow{d} (X, L(X), Y, C, X \circ C, Z, L(Z)) \quad \text{as } n \to \infty.$$

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2. Proofs: structure and ideas

The Septuple Limit Theorem follows directly from the Main Limit Theorem [MPUB25], stability of time-change equations for CBIs [CPGUB13] and the invariance principle for Local times [MUB22] applied to the downward skip-free random walk X_1.

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- ◇ The proof of Main Limit Theorem presents a new **paradigm** for the verification of the assumptions of the invariance principle for local times [MUB22] that might have applications in other settings (reinforced RWs, RWs in random environment, etc).
- ◇ The verification of the assumptions in [MUB22] for BGWI Z_1 is non-trivial, requiring novel results on branching processes and their scaling limits.

Invariance principle for local times [MUB22] in the language of BGWIs

Theorem ([MUB22, Thm 1])

Assumption (SL) holds with $\delta = \frac{d}{\alpha c} \in (0,1)$ *and* $b_n^\alpha \sim n\ell(1/b_n)$ *as* $n \to \infty$. *Let sequence of scaled BGWIs* $(Z_n \overset{d}{=} Z_1(\lfloor n\cdot \rfloor)/b_n)$ *be coupled with self-similar CBI* Z. *For any* $t \in [0,\infty)$, *let*

$$g_t(Z) := \sup\{s \le t : Z(s) = 0\} \quad \textit{and} \quad d_t(Z) := \inf\{s > t : Z(s) = 0\},$$

and the corresponding $g_t(Z_n)$ *and* $d_t(Z_n)$. *Assume* $Z_n \xrightarrow{\mathbb{P}} Z$ *and, for every* $t > 0$,

$$\boldsymbol{g_t(Z_n) \xrightarrow{\mathbb{P}} g_t(Z)} \quad \textit{and} \quad \boldsymbol{d_t(Z_n) \xrightarrow{\mathbb{P}} d_t(Z)}, \textit{ as } n \to \infty. \tag{2}$$

Then for $L_n(t) := |\{s \in \mathbb{N}/n : Z_n(s) = 0\} \cap [0,t]|$, *some* $\tilde{\kappa} > 0$ *and* $\tilde{c}_n := \frac{\tilde{\kappa}}{\mathbb{P}\left(d_{1/n}(Z_n) > 1\right)}$, *the following limit in probability holds:*

$$(Z_n, L_n/\tilde{c}_n) \xrightarrow{\mathbb{P}} (Z, L) \textit{ as } n \to \infty.$$

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Task: for a coupling (Z_n, Z), prove (2) and show $\tilde{c}_n \sim c_n = \kappa n \mathbb{P}(Z_1(n) = 0)$ as $n \to \infty$.

Proving $g_t(Z_n) \xrightarrow{\mathbb{P}} g_t(Z)$ and $d_t(Z_n) \xrightarrow{\mathbb{P}} d_t(Z)$ for examples in [MUB22]

The applications in [MUB22] of the general form of Theorem on the previous slide for regenerative processes use path-wise arguments on the Skorokhod space to prove

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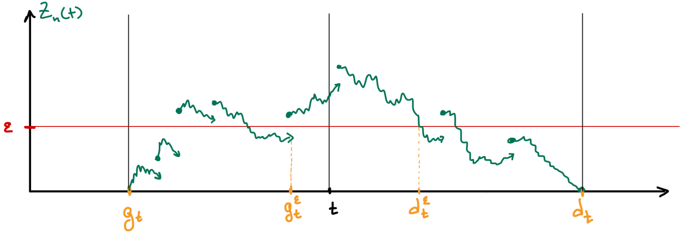
A different approach is required for BGWIs!!

Proving $g_t(Z_n) \xrightarrow{\mathbb{P}} g_t(Z)$ and $d_t(Z_n) \xrightarrow{\mathbb{P}} d_t(Z)$ for branching processes

For $\varepsilon > 0$, for excursions of Z and Z_n straddling time $t > 0$, set

$$g_t^\varepsilon(Z) := \sup\{s \le t : Z(t) \in [0,\varepsilon)]\} \quad \text{and} \quad d_t^\varepsilon(Z) := \inf\{s > t : Z(t) \in [0,\varepsilon)\}\,.$$



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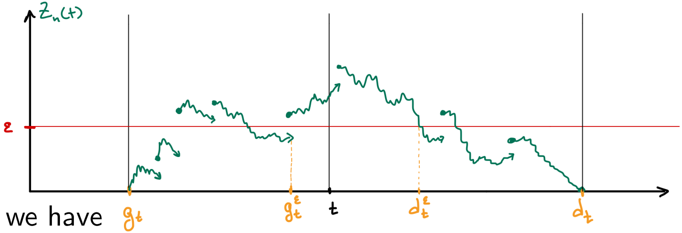

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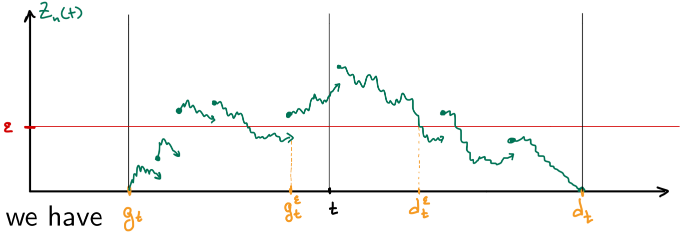

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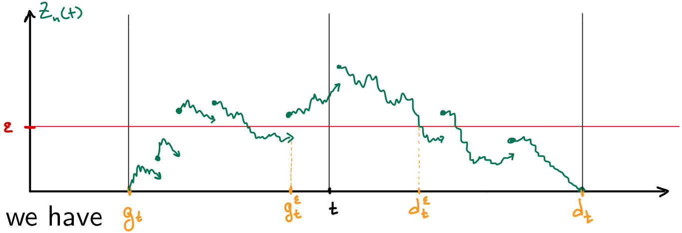

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We need to prove

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} P(n,\varepsilon) = 0, \quad \text{where} \quad P(n,\varepsilon) \in \{\mathrm{I}_d, \mathrm{II}_d, \mathrm{III}_d, \mathrm{I}_g, \mathrm{II}_g, \mathrm{III}_g\}.$$

Proofs of $\mathrm{I}_g, \mathrm{I}_d, \mathrm{II}_g, \mathrm{II}_d$ use path-wise arguments only

$$\mathrm{I}_d = \mathbb{P}(|d_t(Z) - d_t^\varepsilon(Z)| > \eta/3) \xrightarrow{\varepsilon\downarrow 0} 0 \quad \& \quad \mathrm{I}_g = \mathbb{P}(|g_t(Z) - g_t^\varepsilon(Z)| > \eta/3) \xrightarrow{\varepsilon\downarrow 0} 0$$

require **quasi left continuity** of CBI Z and that $\boldsymbol{Z(s-) > 0}$ for any $\boldsymbol{g_t(Z) < s < d_t(Z)}$.

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Lemma

If $f \in D$ satisfies $T_O(f) \le \min\{T_C(f), T_C^-(f)\}$, where $O = [0,\varepsilon)$, $C = [0,\varepsilon]$ and $T_O = \inf\{t \ge 0 : f(t) \in O\}$, $T_C^- = \inf\{t \ge 0 : f(t-) \in C\}$, then for any $f_n \to f$ we have

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II_g: time reverse $\hat{Z}(s) := Z(\max\{t-s, 0\}-)$, note that $g_t^\varepsilon(Z_n) = t - T_O(\hat{Z}_n)$ and $g_t^\varepsilon(Z) = t - T_O(\hat{Z})$ and apply Lemma to $\hat{Z}_n \to \hat{Z}$ to get $|g_t^\varepsilon(Z) - g_t^\varepsilon(Z_n)| \xrightarrow{\text{a.s.}} 0$ as $n \uparrow \infty$

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Theorem (Theorem 6 in [MPUB25])

Under Assumption (SL) with $\delta = \frac{d}{\alpha c} \in (0,1)$, for any $t \geq 0$ and $\varepsilon > 0$, we have

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By [MPUB25, Thm 6] we get

$$g_t^\varepsilon(Z_n) - g_t(Z_n) \xrightarrow{d} g_t^\varepsilon(Z) - g_t(Z) \quad \text{and} \quad d_t(Z_n) - d_t^\varepsilon(Z_n) \xrightarrow{d} d_t(Z) - d_t^\varepsilon(Z) \quad \text{as } n \to \infty$$

and hence $\mathrm{III}_d = \mathbb{P}(|d_t(Z_n) - d_t^\varepsilon(Z_n)| > \eta/3) \xrightarrow{n\uparrow\infty} \mathbb{P}(|d_t(Z) - d_t^\varepsilon(Z)| > \eta/3) = \mathrm{I}_d$. Ditto for $\mathrm{III}_g \xrightarrow{n\uparrow\infty} \mathrm{I}_g$.

$\mathrm{III}_d = \mathbb{P}(|d_t(Z_n) - d_t^\varepsilon(Z_n)| > \eta/3)$ & $\mathrm{III}_g = \mathbb{P}(|g_t(Z_n) - g_t^\varepsilon(Z_n)| > \eta/3)$

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$$(g_t^\varepsilon(Z_n), g_t(Z_n)) \xrightarrow{d} (g_t^\varepsilon(Z), g_t(Z)) \quad \textit{and} \quad (d_t^\varepsilon(Z_n), d_t(Z_n)) \xrightarrow{d} (d_t^\varepsilon(Z), d_t(Z)) \quad \textit{as } n \to \infty.$$

By [MPUB25, Thm 6] we get

$$g_t^\varepsilon(Z_n) - g_t(Z_n) \xrightarrow{d} g_t^\varepsilon(Z) - g_t(Z) \quad \text{and} \quad d_t(Z_n) - d_t^\varepsilon(Z_n) \xrightarrow{d} d_t(Z) - d_t^\varepsilon(Z) \quad \text{as } n \to \infty$$

and hence $\mathrm{III}_d = \mathbb{P}(|d_t(Z_n) - d_t^\varepsilon(Z_n)| > \eta/3) \xrightarrow{n\uparrow\infty} \mathbb{P}(|d_t(Z) - d_t^\varepsilon(Z)| > \eta/3) = \mathrm{I}_d$. Ditto for $\mathrm{III}_g \xrightarrow{n\uparrow\infty} \mathrm{I}_g$. We already know $\mathrm{I}_d, \mathrm{I}_g \xrightarrow{\varepsilon\downarrow\infty} 0$ finishing the proof.

$\mathrm{III}_d = \mathbb{P}(|d_t(Z_n) - d_t^\varepsilon(Z_n)| > \eta/3)$ & $\mathrm{III}_g = \mathbb{P}(|g_t(Z_n) - g_t^\varepsilon(Z_n)| > \eta/3)$

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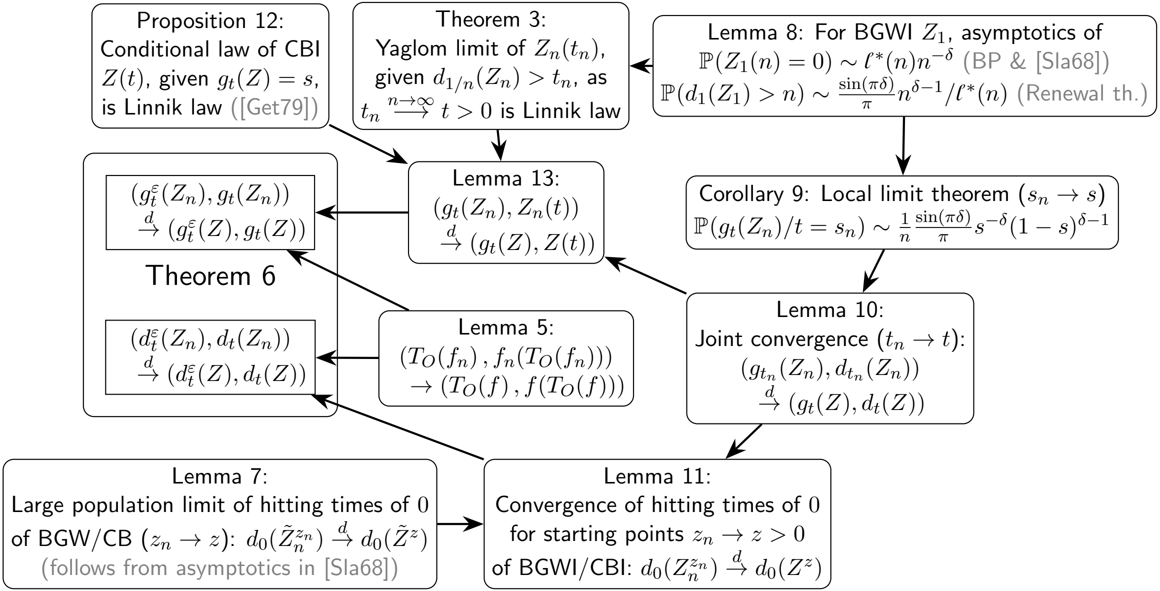
By [MPUB25, Thm 6] we get

$$g_t^\varepsilon(Z_n) - g_t(Z_n) \xrightarrow{d} g_t^\varepsilon(Z) - g_t(Z) \quad \text{and} \quad d_t(Z_n) - d_t^\varepsilon(Z_n) \xrightarrow{d} d_t(Z) - d_t^\varepsilon(Z) \quad \text{as } n \to \infty$$

and hence $\mathrm{III}_d = \mathbb{P}(|d_t(Z_n) - d_t^\varepsilon(Z_n)| > \eta/3) \xrightarrow{n\uparrow\infty} \mathbb{P}(|d_t(Z) - d_t^\varepsilon(Z)| > \eta/3) = \mathrm{I}_d$. Ditto for $\mathrm{III}_g \xrightarrow{n\uparrow\infty} \mathrm{I}_g$. We already know $\mathrm{I}_d, \mathrm{I}_g \xrightarrow{\varepsilon\downarrow\infty} 0$ finishing the proof.

But how does one prove [MPUB25, Thm 6]?



(numbers next to lemmas, corollary, proposition and theorems correspond to those in the preprint [MPUB25])

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Thank you for your attention!!

[MPUB25] available on arXiv:2503.20923

References

- Ma. Emilia Caballero, José Luis Pérez Garmendia, and Gerónimo Uribe Bravo, *A Lamperti-type representation of continuous-state branching processes with immigration*, Ann. Probab. **41** (2013), no. 3A, 1585–1627.
- Clément Foucart and Gerónimo Uribe Bravo, *Local extinction in continuous-state branching processes with immigration*, Bernoulli **20** (2014), no. 4, 1819–1844.
- R. K. Getoor, *Excursions of a Markov process*, Ann. Probab. **7** (1979), no. 2, 244–266.
- Zenghu Li, *A limit theorem for discrete Galton-Watson branching processes with immigration*, J. Appl. Probab. **43** (2006), no. 1, 289–295.
- A. Mijatović, B. Povar, and G. Uribe Bravo, *Critical branching processes with immigration: scaling limits of local extinction sets*, arXiv:2503.20923 (2025).
- Aleksandar Mijatović and Gerónimo Uribe Bravo, *Limit theorems for local times and applications to SDEs with jumps*, Stochastic Process. Appl. **153** (2022), 39–56.
- R. S. Slack, *A branching process with mean one and possibly infinite variance*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **9** (1968), 139–145.