INSTITUT de MATHEMATIQUES de TOULOUSE

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A fragmentation toy-model for the perimeter of the convex hull of a Brownian motion in a disk

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joint work with Bénédicte Haas

IMT – Université de Toulouse

Branching and Persistence
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Convex hull of the Brownian motion

Proposition

Let B be a Brownian motion reflected at the boundary of the unit disk in $\mathbb{R}^2$. Write H_t for convex hull of $(B_s, 0 \leq s \leq t)$, we have

$$P_t := \mathrm{Perimeter}(H_t) = \int_0^{2\pi} \left(\sup_{0 \leq s \leq t} (\theta \cdot B_s) \right) \mathrm{d}\theta$$

Question

Can we study the asymptotic properties of P_t as $t \to \infty$?

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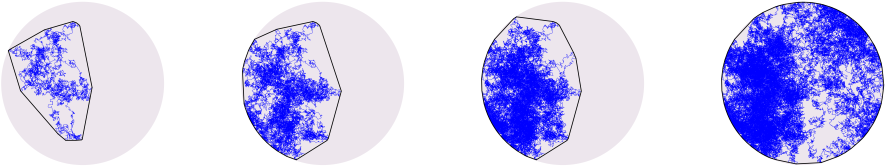
Convergence of the perimeter



(a) Time $t = 1$ (b) Time $t = 4$ (c) Time $t = 8$ (d) Time $t = 16$

Figure: Convex hull (in black) of the trajectory of the Brownian motion B (in blue) in the unit disk over the time interval $[0, t]$. We observe that $P_t \to 2\pi$ as $t \to \infty$.

Outline

1. First moment estimate for the convex hull
2. Reduction to a toy-model
3. Analysis of the fragmentation process

Origin of the project

Schedule

09:20-10:00 **Sandro Franceschi**
Reflected Brownian motion in a cone: study of the transient case. Escape and absorption probability, Green's functions and Martin's boundary.

10:10-10:50 **Grégory Schehr**
Statistics of the maximum and the convex hull of a Brownian motion in confined geometries

Coffee break

11:20-12:00 **Vitali Wachtel**
Supermartingale approach to random walks in cones

First moment asymptotic

"Theorem" (De Bruyne et al '22)

There exists $c > 0$ such that

$$2\pi - \mathbb{E}(P_t) \sim_{t\to\infty} ct^{1/4} \exp\left(-2t^{1/2}\right).$$

First moment asymptotic
Proof 1/2

Using Cauchy's formula and invariance by rotation, we have for all $\phi \in \mathbb{S}$

$$\mathbb{E}(P_t) = \mathbb{E}\left(\int_0^{2\pi} \sup_{0\leq s\leq t} (\theta \cdot B_s) \mathrm{d}\theta\right) = 2\pi \mathbb{E}\left(\sup_{0\leq s\leq t} (\phi \cdot B_s)\right),$$

thus

$$\mathbb{E}(P_t) = 2\pi \int_0^1 \mathbb{P}\left(\sup_{0\leq s\leq t} (\phi \cdot B_s) \geq x\right) \mathrm{d}x.$$

Writing $\tau_x := \inf\{t > 0 : B_t \cdot \phi \geq x\}$, we have

$$2\pi - \mathbb{E}(P_t) = 2\pi \int_0^1 \mathbb{P}(\tau_x > t) \mathrm{d}x.$$

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This is a famous problem, consisting in estimating the probability of escape of Brownian motion from the disk through a narrow gap.

Property/Conjecture

As $x \to 1$, we have

$$\mathbb{E}(\tau_x) = -\log(1-x) + O(1).$$

Moreover, one expects that for x close enough to 1 and t large enough,

$$\mathbb{P}(\tau_x > t) \approx_{t\to\infty} \exp(-t/\mathbb{E}(\tau_x)) \approx \exp\left(-\frac{t}{-\log(1-x)}\right)$$

First moment asymptotic

Proof 2/2

Coming back to the previous equation, we have

$$
\begin{aligned}
2\pi - \mathbb{E}(P_t) &= 2\pi \int_0^1 \mathbb{P}(\tau_x > t)\mathrm{d}x \\
&\approx 2\pi \int_0^1 \exp\left(-\frac{t}{-\log(1-x)}\right)\mathrm{d}x = 2\sqrt{t}K_1(2\sqrt{t}),
\end{aligned}
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where K_1 is the modified Bessel function of the first kind. Using that

$$K_1(x) \sim \sqrt{\frac{\pi}{2x}}e^{-x},$$

the proof is now complete.

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Outline

Reduction to a toy-model

Recall that $\mathbb{E}(P_t) = 2\pi - ct^{1/4}e^{-2t^{1/2}}(1 + o(1))$.

- We aim to explain this asymptotic behaviour using a toy-model, especially the semi-exponential decay.
- The physicists computation crucially depends on the narrow escape distribution approximation by an Exponential variable.
- We construct a model approaching the distribution of the perimeter of the Brownian motion making the replacement.

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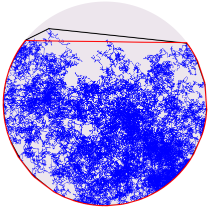
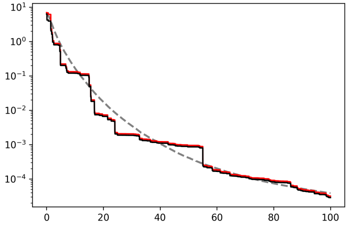
Reduced perimeter of the Brownian motion

Recall that H_t is the convex hull of $\{B_s, 0 \leq s \leq t\}$. We write $\widetilde{H}_t$ the convex hull of $\{B_s, 0 \leq s \leq t\} \cap \mathbb{S}$, we have

$$\widetilde{P}_t := \text{Perimeter}(\widetilde{H}_t) \approx P_t \quad \text{for } t \text{ large enough.}$$



(a) Visual comparison of the hulls $\widetilde{H}_t$ (in red) and H_t (in black).



(b) Distance to 2π of the perimeters of H and $\widetilde{H}$. One graph has been slightly raised for visibility.

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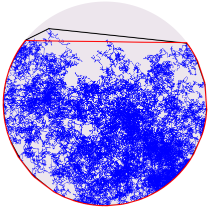

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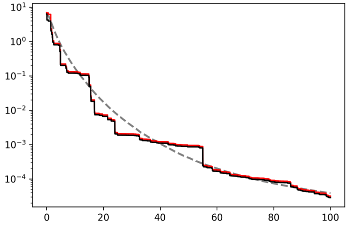

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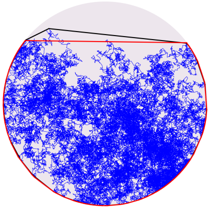

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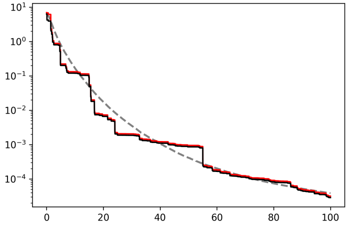

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Fragmentation description of the reduced perimeter 1/3

Definition

We write $(\ell_j(t), j \geq 1)$ the list of lengths of the intervals in $\mathbb{S}\backslash\{B_s, 0 \leq s \leq t\}$.

Property

Recall that $\widetilde{P}_t = \mathrm{Perimeter}(\widetilde{H}_t)$ and write $\widetilde{A}_t = \mathrm{Area}(\widetilde{H}_t)$. As $t \to \infty$,

$$2\pi - \widetilde{P}_t = \sum_{i \geq 1} (\ell_i(t) - 2\sin(\ell_i(t)/2)) \sim \frac{1}{24} \sum \ell_i(t)^3,$$

$$\pi - \widetilde{A}_t = \frac{1}{2} \sum_{i \geq 1} (\ell_i(t) - \sin(\ell_i(t))) \sim \frac{1}{12} \sum \ell_i(t)^3,$$

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Fragmentation description of the reduced perimeter 2/3

Some observations:

- The sequence $(\ell_i(t), t \geq 0)$ evolves by jumps at times $\{t \in \mathbb{R}_+ : \|B_t\| = 1\}$.
- At each such time t, the interval J containing B_t is split into two sub-intervals J_1, J_2, the length of the sub-segments being equal to the parent segment.
- The segment of length $\ell_i(t)$ splits at a time obtained by the narrow escape problem, approached by an Exponential random variable with parameter $(2 \log \ell_i(t))^{-1}$.

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We suggest the following approximation by a fragmentation process $(x_i(t), t \geq 0)$. Let $c \geq 0$, ν a probability distribution on $\{(p_j) \in [0,1]^{\mathbb{N}\downarrow} : \sum p_j = 1\}$.

- It starts from the initial condition $(1, 0, 0, 0, \ldots)$.
- Each element i of length $x_i(t)$ splits after an exponential time of parameter $\frac{1}{2 \log x_i(t)}$.
- A splitting particle of mass x is replaced by particles of mass $xp_1, xp_2, \ldots$, with (p_j) chosen according to the law ν.
- Additionally, each element *melds* at rate c, i.e. a particle of mass x at time 0 has time mass xe^{-ct} at time t, provided it did not split.

We have to be a little bit creative at time $t = 0$ where the particle splits at infinite rate. No theoretical (or practical) difficulty there.

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Outline

Evolution of the *typical fragment*

Let $c \geq 0$, ν a probability distribution on $\{\mathbf{p} \in [0,1]^{\mathbb{N}\downarrow} : \sum p_j = 1\}$.

Proposition

Let ξ be a Lévy process with Laplace exponent

$$-\log \mathbb{E}(e^{-q\xi_1}) =: \phi(q) = q(c+1) + \int \left(1 - \sum_{j \geq 1} p_j^q\right) \nu(\mathrm{d}\mathbf{p}).$$

Set $\rho(t) := \inf\{u \geq 0 : \int_0^u \xi_r \mathrm{d}r \geq t\}$. For any measurable bounded function f,

$$\mathbb{E}\left(\sum_{j \geq 1} x_i(t) f(x_i(t))\right) = \mathbb{E}\left(f\left(\exp\left(-\xi_{\rho(t)}\right)\right)\right).$$

Laplace transform of the typical fragment

Theorem (Laplace transfom of the typical fragment, Haas–M.)

Let $\Phi : q \mapsto \int_0^q \phi(s)\mathrm{d}s$. It is the Laplace transform of a (spectrally negative) Lévy process, and Φ^{-1} its inverse is the Laplace exponent of a subordinator, with Lévy measure L. For all $q > 0$ and $t \geq 0$, we have

$$\mathbb{E}\left(e^{-q\xi_{\rho(t)}}\right) = \phi(q) \int_0^\infty e^{-\Phi(q)x - \frac{t}{x}} x L(\mathrm{d}x).$$

- There is a surprizing connection between ξ_ρ and the law of a Lévy process X of law Φ. Is there a nice trajectorial relationship between ξ and X?
- *In particular, for $q = 3$, the Laplace transform of $\xi_{\rho(t)}$ gives the analogue of the expected perimeter for our toy-model.*

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We remark that the function

$$f : (t, q) \mapsto (\Phi^{-1})'(q)\mathbb{E}\left(e^{-\Phi^{-1}(q)\xi_{\rho(t)}}\right)$$

is a strong solution to the equation $\partial_t\partial_q f = \partial_q\partial_t f = f$, with $f(0, q) = \int xe^{-qx}L(\mathrm{d}x)$, using that, by change of variable

$$f(t, q) = \int_t^\infty \mathbb{E}\left(\frac{e^{-\Phi^{-1}(q)\xi_{\rho(s)}}}{\xi_{\rho(s)}}\right)\mathrm{d}s.$$

We solve this partial differential equation, remarking that

$$\int_0^\infty e^{-\lambda t} f(t, q)\mathrm{d}t = \int_0^\infty \int_0^\infty e^{-qx-\frac{t}{x}} xL(\mathrm{d}x)e^{-\lambda t}\mathrm{d}t,$$

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Elements of proof

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Asymptotic behaviour of the Laplace transform

Theorem (Asymptotic behaviour, Haas–M.)

Assume that ϕ is regularly varying at 0 with index $\gamma \in (0,1]$ (and some technical additional condition). There exists an explicit $b(q) > 0$ such that

$$\mathbb{E}\left(e^{-q\xi_{\rho(t)}}\right) \sim_{t\to\infty} b(q) t^{1/4} \Phi^{-1}(t^{-1/2}) e^{-2\sqrt{\Phi(q)t}}.$$

- Note that $\Phi(q)$ is regularly varying with index $\gamma + 1$, and Φ^{-1} with index $\frac{1}{\gamma+1}$.
  Therefore $\mathbb{E}\left(e^{-q\xi_{\rho(t)}}\right) \sim_{t\to\infty} t^{\alpha+o(1)} e^{-2\sqrt{\Phi(q)t}}$ with $\alpha = \frac{1}{4} - \frac{1}{2(\gamma+1)} \in [-\frac{1}{4}, 0]$.
- We recover the main semi-exponential decay from the physicists, but not the $t^{1/4}$ polynomial term.
- The proof relies on applying the Laplace "saddle-point" method on the previous formula, using that L is reasonably regular under our assumptions.

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  Therefore $\mathbb{E}\left(e^{-q\xi_{\rho(t)}}\right) \sim_{t\to\infty} t^{\alpha+o(1)} e^{-2\sqrt{\Phi(q)t}}$ with $\alpha = \frac{1}{4} - \frac{1}{2(\gamma+1)} \in [-\frac{1}{4}, 0]$.
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Long-time behaviour of the fragmentation process

Theorem (Long-time behaviour (Haas–M.))

Let Z_γ be a random variable with Laplace transform

$$\frac{2q^{\gamma/2}}{\Gamma(\frac{\gamma}{\gamma+1})} K_{\frac{\gamma}{\gamma+1}}(2q^{\frac{\gamma+1}{2}}),$$

with K_α the modified Bessel function of the second kind. Under the assumptions of the previous theorem

1. $\lim_{t\to\infty} \Phi^{-1}(1/t)\xi_{\rho(t)} \to Z_\gamma$ *in law,*
2. $\lim_{t\to\infty} \sum_{j\geq 1} x_j(t)\delta_{\Phi^{-1}(1/t)\log x_j(t)} = \mathcal{L}(Z_\gamma)$ *in probability for the topology of weak convergence.*

Related fragmentation models

Homogeneous fragmentation: Particles split at constant rate, independently of one another. At large time t, particles are typically of size e^{-vt} for some $v > 0$ (see Bertoin, Berestycki, Krell, ...).

Self-similar fragmentation: Particle of mass x splits at rate x^{α}.

- $\alpha > 0$ Particles are typically of size $t^{-1/\alpha}$ at large times (see Bertoin, ...)
- $\alpha < 0$ Fragmentation into dust in finite time (see Bertoin, Goldschmidt, Haas, Rivero, ...)

Branching-fragmentation: particle of mass x splits at rate $|\log x|$. The typical fragment evolves according to a $-\log$ CSBP. Not yet studied?

Our model is at the interface between self-similar process with $\alpha > 0$ and homogeneous fragmentation.

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Self-criticism

Exponential approximation of the narrow escape This is a classical assumption, with robust numerical justification. It is hard to believe the discrepancy in polynomial orders could come from this choice. Using multitype versions of this model could allow a more general class of splitting times.

Independence between splitting times It should be clear that two nearby intervals will typically be split around the same time by Brownian motion. However, for far apart fragments (measured in term of their size) the assumption is rather natural. The assumption also seems natural, especially given the convergence to a deterministic limit of the process.

Choice of the fragmentation measure Zooming around a typical small fragment, we should see a Brownian motion in the *half-plane*, which is recurrent. The law ν should be given by the fragmentation of $[-1/2, 1/2]$ by the Brownian motion W on the half-plane between time 0 and $T = \inf\{t > 0 : \|W_t\| > R\}$, with R being the distance at which the approximation of the disk by the half-plane is no longer relevant.

A natural extension of the toy-model that would provide a better approximation for our model of interest is a fragmentation model with *time-inhomogeneous* fragmentation distribution.

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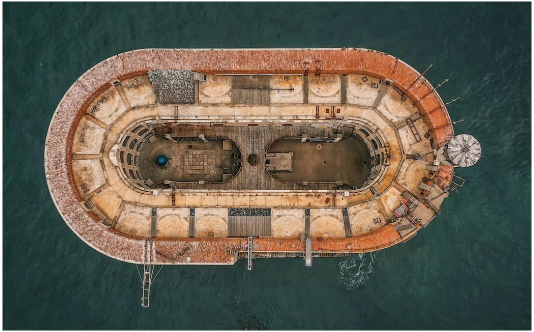
Future directions of work

1. Generalization of the fragmentation model to splitting rate given by $\log x^{-k}$ or more generally to a regularly varying function in $\log x$.
2. Direct study of the fragmentation of the unit disk by the Brownian motion, taking correlations in consideration.
3. Generalisation of these computations to higher dimensions.

Conclusion



Thank you for your attention!