Continuous Branching process with Immigration conditioned to stay positive

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- BGWI (Bienaymé-Galton-Watson process with immigration) is described by two generating functions f and g, where f corresponds to regeneration and g corresponds to immigration.
- Under specific scalings (and possibly considering a sequence of processes) BGWIs are known to converge to Continuous Branching processes with immigration (CBI), see [Kawazu and Watanabe 1971].

BGWI and CBI processes

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- In [Caballero, Pérez Garmendia, and Uribe Bravo 2013] it was shown that a CBI process Z is a solution to a time-change equation:

$$Z_t = X\Big(\int_0^t Z_s ds\Big) + Y_t, \tag{1}$$

where Y is a subordinator and X is a spectrally positive Levy process with finite mean.

Consider the generating functions

$$f(s) := s + c(1-s)^{\alpha+1}L(1-s) \quad \text{and} \quad g(s) := 1 - d(1-s)^{\alpha}G(1-s),$$

where $c,\ d > 0$ and $\alpha \in (0,1]$ and $L(x) \sim G(x)$ for $x \to 0$ and L is slowly varying at 0.

Note that the $\mathrm{BGWI}(f,g)$ is critical: $f'(1) = 1$, and

$$f''(1) < \infty \quad \text{iff} \quad g'(1) < \infty \quad \text{iff} \quad \alpha = 1 \text{ and } G(0) < \infty.$$

Proposition

Let η_α be a r.v. with a tail $\mathbb{P}(\eta_\alpha > x) \sim dx^{-\alpha}G(x)$ as $x \to \infty$. Then

$$g \sim Pois(\eta_\alpha) \quad \text{and} \quad f \sim Be(1/Pois(\eta_\alpha)) \cdot (Pois(\eta_\alpha) + 1).$$

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The scaling limit

Denote by $\mathfrak{z}$ the $\mathrm{BGWI}(f, g)$. Define b_n by $b_n^\alpha \sim nL(1/b_n)$ (as $n \to \infty$) and let

$$Z_t^{(n)} := \frac{1}{b_n} \mathfrak{z}(\lfloor nt \rfloor). \quad (2)$$

It holds that (in Skorokhod's space)

$$Z^{(n)} \xrightarrow{w} Z, \; n \to \infty,$$

where Z is an α-self-similar CBI: the regenerating and immigration mechanisms are given by

$$R(q) = -cq^{\alpha+1} \quad \text{and} \quad F(q) = dq^\alpha;$$

or, equivalently, Z satisfies the time-change equation $Z_t = X\left(\int_0^t Z_s ds\right) + Y_t$ with Y being an α-stable subordinator and X being a $1 + \alpha$-stable spectrally positive Levy process.

We assume that

$$\delta := \frac{d}{\alpha c} \in (0, 1). \quad (3)$$

This assumption guarantees that 0 is a recurrent point of Z, [Foucart and Uribe Bravo 2014].

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Plan

1. Yaglom limit: conditioning on non-extinction up till time $u > 0$.
2. Durrett's extension to the meander.
3. Letting $u \to \infty$.
4. Recognising the h-transform of the original process.

Yaglom limit

Let $d_0(w)$ denote the hitting time of 0 for w:

$$d_0(w) \coloneqq \inf\{s > 0 : w(s) = 0\}.$$

Consider

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{1}{b_n}\mathfrak{z}(n) \in A \,\middle|\, d_0(\mathfrak{z}) > n \right).$$

Theorem (Yaglom limit, [Mijatović, P., and Uribe Bravo 2025])

$$\lim_{n\to\infty} \mathbb{E}\left(\exp\left(-\lambda \frac{\mathfrak{z}(n)}{b_n} \right) \,\middle|\, d_0(\mathfrak{z}) > n \right) = \frac{1}{(1+\alpha c \lambda^\alpha)}.$$

Compare to the similar limit for a BGW $\tilde{\mathfrak{z}}$, [Slack 1968]:

$$\lim_{n\to\infty} \mathbb{E}\left(\exp\left(-\lambda \frac{\tilde{\mathfrak{z}}(n)}{b_n} \right) \,\middle|\, d_0(\tilde{\mathfrak{z}}) > n \right) = 1 - \left(\frac{\alpha c \lambda^\alpha}{1+\alpha c \lambda^\alpha} \right)^{1/\alpha}.$$

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Yaglom limit extension to the whole excursion, general case

Consider a Markov chain $(v(k))_{k\geq 0}$ on $\mathbb{R}$ and assume that

1. Scaling limit: for $(b_n)_n$ and $z_n := v(0)/b_n \to z$, it holds $Z_n^{z_n} := v(n\cdot)/b_n \xrightarrow{w} Z^z$.
2. Enough mass beyond 0: for $t, z > 0$, $\mathbb{P}(\inf_{s<t} Z^z(t) > 0) > 0$.
3. Convergence of hitting times of 0: $t_n \to t > 0$ and $z_n \to z > 0$, then $\mathbb{P}(d_0(Z_n^{z_n}) > t_n) \to \mathbb{P}(d_0(Z^z) > t)$.
4. Regularity of 0: $t_n \to t > 0$ and $z_n \to 0$, then $\mathbb{P}(d_0(Z_n^{z_n}) > t_n) \to 0$.

Theorem ([Durrett 1978])**

*Assume **1-4** hold. Fix $u > 0$ and let $Z_n^{(+,u)} := (Z_n^0 \mid d_0(Z_n) > u)$. Then*

1. *$\exists \beta > 0$ such that $\mathbb{P}(d_0(Z_n) > u) \sim n^{-\beta} L^*(n)$ for a slowly varying L^*;*
2. *$(Z_n^{(+,u)}(s))_{s\in[0,u]}$ converges weakly (in $\mathscr{D}[0,\infty)$) to an inhomogeneous Markov process $Z^{(+,u)}$.*

Yaglom limit extension to the whole excursion, general case

Define $\boldsymbol{\kappa_u}$ as $Z_n^{(+,u)}(u) \xrightarrow{w} \boldsymbol{\kappa_u}$. By the self-similarlity of the limit, this will satisfy $\boldsymbol{\kappa_u} := u^{1/\alpha}\boldsymbol{\kappa_1}$. The transition density of $Z^{(+,u)}$ is given by:

for $t > 0$

$$\mathbb{P}_0(Z^{(+,u)}(t) \in dy) = (u/t)^{\beta} \; \mathbb{P}((t/u)^{1/\alpha}\boldsymbol{\kappa_u} \in dy) \; \mathbb{P}_z \left(d_0(Z) > u - t\right);$$

for $s < t < u$ and $z > 0$

$$\begin{aligned} &\mathbb{P}(Z^{(+,u)}(t) \in dy \mid Z^{(+,u)}(s) = z) \\ &\qquad = \mathbb{P}_z \left(\begin{array}{c} Z(t-s) \in dy, \\ d_0(Z) > t - s \end{array} \right) \frac{\mathbb{P}_y(d_0(Z) > u - t)}{\mathbb{P}_z(d_0(Z) > u - s)}. \end{aligned}$$

Yaglom limit extension to the whole excursion, BGWI case

Theorem

Consider $Z_n^{(+,u)} := (\mathfrak{z}(\lfloor n\cdot \rfloor)/b_n \mid d_0(\mathfrak{z}) > nu)$. *There exists an inhomogeneous Markov process* $Z^{(+,u)}$ *such that*

$$Z_n^{(+,u)} \xrightarrow{w} Z^{(+,u)}, \text{ as } n \to \infty.$$

Constant β is given by [Mijatović, P., and Uribe Bravo 2025] and it is equal to $\beta := 1 - \delta$.
From Yaglom limit we find that κ_u is defined as

$$\mathbb{E}e^{\lambda \kappa_u} = \frac{1}{(1 + \alpha c u \lambda^\alpha)}.$$

Define a random variable κ as $\mathbb{E}e^{-\lambda\kappa} = (1 + \lambda^\alpha)^{-1}$. Then

$$\mathbb{P}_0(Z^{(+,u)}(t) \in dy) = (u/t)^{1-\delta}\, \mathbb{P}((\alpha c t)^{1/\alpha}\kappa \in dy)\, \mathbb{P}_y(d_0(Z) > u - t).$$

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Letting $u \to \infty$: part I

For $s < t < u$ and $z > 0$

$$\lim_{u\to\infty} \mathbb{P}(Z^{(+,u)}(t) \in dy \mid Z^{(+,u)}(s) = z)$$
$$= \mathbb{P}_z \begin{pmatrix} Z(t-s) \in dy, \\ d_0(Z) > t-s \end{pmatrix} \lim_{u\to\infty} \frac{\mathbb{P}_y(d_0(Z) > u-t)}{\mathbb{P}_z(d_0(Z) > u-s)}.$$

Proposition (by-product of [Mijatović, P., and Uribe Bravo 2025])

Let $A \sim Arcsin(\delta) \sim Beta(\delta, 1-\delta)$ and independent $E \sim Exp(1)$. Then

$$d_0(Z^z) \overset{d}{=} \frac{z^\alpha}{\alpha c} \frac{E^{-\alpha}}{A}.$$

For $h(z) = z^{\alpha(1-\delta)}$, we have

$$\mathbb{P}_z(d_0(Z) > u) \sim \frac{\Gamma(1-\alpha)}{\Gamma(\delta)\Gamma(2-\delta)} \frac{h(z)}{(\alpha c u)^{1-\delta}}, \quad \text{as } u \to \infty.$$

Hence

$$\mathbb{P}(Z^{(+,u)}(t) \in dy \mid Z^{(+,u)}(s) = z) \to \mathbb{P}_z(Z(t-s) \in dy, d_0(Z) > t-s)\, h(y)/h(z).$$

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Theorem

There exists a (time-homogeneous) Markov process $Z^{(+)}$, strictly positive on $(0, \infty)$, such that the finite-dimensional distributions of $Z^{(+,u)}$ converge to those of $Z^{(+)}$ as $u \to \infty$:

$$Z^{(+,u)} \stackrel{fdd}{\to} Z^{(+)}, \quad \text{as } u \to \infty. \tag{4}$$

The process $Z^{(+)}$ is an h-transform of Z with $h(z) := z^{\alpha(1-\delta)}$ and it has the following transition densities:
for $y > 0$, $z > 0$, $0 < t < u$

$$\mathbb{P}_z(Z^{(+)}(t) \in dy) = \mathbb{P}_z(Z(t) \in dy, d_0(Z) > t)\,\frac{h(y)}{h(z)}; \tag{5}$$

for $y > 0$ and $z > 0$ and $0 < s < t$

$$\mathbb{P}(Z^{(+)}(t) \in dy \,|\, Z^{(+)}(s) \in dz) = \mathbb{P}_z(Z^{(+)}(t-s) \in dy);$$

for $z = 0$ and $t > 0$

$$\mathbb{P}_0(Z^{(+)}(t) \in dy) = \frac{\Gamma(1-\alpha)}{\Gamma(\delta)\Gamma(2-\delta)}\,\frac{h(y)}{(\alpha ct)^{1-\delta}}\mathbb{P}((\alpha ct)^{1/\alpha}\kappa \in dy). \tag{6}$$

Letting $u \to \infty$: h-transform

To see that the function h is a Doob h-transform we need to verify that it is excessive for the process killed at the hitting time of 0, i.e., $\mathbb{E}_z\left[h(Z(t));\, d_0(Z) > t\right] \uparrow h(z)$ as $t \downarrow 0.$, we note that for any $z > 0$

$$\mathbb{E}_z\left[\frac{h(Z(t))}{h(z)};\, d_0(Z) > t\right] = \mathbb{P}_z^{(+)}\left[1\right] \leq 1, \tag{7}$$

where $\mathbb{P}_z^{(+)}$ is the probability measure associated with $Z^{(+)}$ issued from z. As $t \downarrow 0$, by Lebesgue's Dominated convergence theorem, we see that $\mathbb{E}_z\left[h(Z(t));\, d_0(Z) > t\right] \uparrow h(z)$ as required.

Consider the Lamperti transform of Z

$$Z(t)1_{\{d_0(Z)>t\}} = ze^{\xi(\tau(tz^{-\alpha}))}, \text{ where } \tau_t := \inf\{s > 0 : \int_0^s e^{\alpha\xi(u)}u > t\}. \quad (8)$$

Since Z hits zero continuously, $\lim_{t\to\infty} \xi_t = -\infty$.

Theorem

The Laplace transform of ξ is given by

$$\Psi(\lambda) := \log\left(\mathbb{E}e^{-\lambda\xi}\right) = \frac{\Gamma(\alpha+\lambda)}{\Gamma(\lambda)}\alpha c(1-\delta+\lambda/\alpha).$$

Moreover, we find that $\mathbb{E}e^{\alpha(1-\delta)\xi} = 1$, i.e., the Cramer's constant of ξ is $\alpha(1-\delta)$ and also $\left(e^{\alpha(1-\delta)\xi_t}\right)_{t\geq 0}$ is a martingale.

Theorem (Volkonskii's theorem)

The generator of Z and the generator of ξ are related through

$$\mathcal{L}_Z f(z) = z^{-\alpha}\mathcal{L}_{e^\xi} f(z) = z^{-\alpha}\mathcal{L}_\xi \tilde{f}(\log(z)),$$

where $\tilde{f}(z) := f(e^z)$.

Being an h-transform implies that $h(Z(t))1_{\{d_0(Z)>t\}}$ is a supermartingale. It is actually a martingale.

Theorem

The process $\left(h(Z(t))1_{\{d_0(Z)>t\}}\right)_t = \left(h(ze^{\xi(\tau(tz^{-\alpha}))})\right)_t$ is a martingale.

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- From [Bertoin and Doney 1994], as $x \to \infty$

$$\mathbb{P}_z(\max_{t \le d_0(Z)} Z(t) > x) = \mathbb{P}(z \max_{t \ge 0} e^{\xi(t)} > x) \sim \frac{C_1/C_2}{\alpha(1-\delta)} \frac{h(z)}{h(x)},$$

  where $C_1 := -\log \mathbb{P}(H_1 < \infty)$ and $C_2 := \mathbb{E}(H_1 e^{\alpha(1-\delta)H_1}; H_1 < \infty)$ with H being the ascending ladder height process of ξ.

- As $x \to \infty$

$$\mathbb{P}_0(Z^{(+)}(t) > x) \sim \frac{\sin(\pi\delta)}{\pi\delta(1-\delta)} (\alpha c t)^{\delta} x^{-\alpha\delta}.$$

Following [Noba 2023], we can define the q-scale function of a spectrally positive process Z as

$$W_Z^{(q)}(x,y) = \begin{cases} 1/N^x \left[e^{-qT_y^-} \right], & y \leq x, \\ 0, & y > x, \end{cases}$$

for x, y in $(0, \infty)$; where N^x is the excursion measure (from x) and T_y^- is the hitting time of the $[0, y]$.

$$\mathbb{E}_x \left[e^{-qT_a^-}; T_a^- < T_b^+ \right] = \frac{W_Z^{(q)}(b,x)}{W_Z^{(q)}(b,a)}$$

Scale function for some spectrally positive processes

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$$\mathbb{E}_x\left[e^{-qT_a^-}; T_a^- < T_b^+\right] = \frac{W_Z^{(q)}(b,x)}{W_Z^{(q)}(b,a)}$$

Scale function for a self-similar CBI

By [Noba 2023], we see that the q-scale function $(W_Z^{(q)}(x,y))_x$ of Z, for $q \geq 0$, is the unique solution to the Volterra integral equation

$$f(x) = x^{\alpha-1} W_\xi(\log(y/x)) + x^{\alpha-1} q \int_{(x,y)} f(x') W_\xi(\log(x'/x)) dx'.$$

In particular, for the 0-scale function of Z is immediately given by

$$W_Z(x,y) = x^{\alpha-1} W_\xi(\log(y/x))$$

Proposition

The 0-scale function of ξ is given by

$$W_\xi(x) = \frac{e^{-\alpha(1-\delta)x}}{\Gamma(1+\alpha)c(1-\delta)} \int_0^{1-e^{-x}} u^{\alpha-1}(1-u)^{-1-\alpha(1-\delta)} u$$
$$= \frac{e^{-\alpha(1-\delta)x}(1-e^{-x})^\alpha}{\Gamma(1+\alpha)\alpha c(1-\delta)} {}_2F_1(\alpha, 1+\alpha(1-\delta); 1+\alpha; 1-e^{-x}).$$

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Scale function of ξ: proof

ξ is a Lévy process hence the scale function satisfies

$$W_\xi^{(q)}(x-y) = W_\xi^{(q)}(x,y) \quad \text{and} \quad \int_0^\infty e^{-\lambda x} W_\xi^{(q)}(x) dx = \frac{1}{\Psi(\lambda) - q}.$$

For $q = 0$, we compute

$$W_\xi(x) = \mathcal{L}^{-1}\left[\frac{1/\Gamma(1+\alpha)}{c(1-\delta)} \frac{\Gamma(\lambda)\Gamma(\alpha)}{\Gamma(\alpha+\lambda)} \frac{\alpha(1-\delta)}{\alpha(1-\delta)+\lambda}\right](x).$$

Finally, recall the Beta function

$$\int_0^\infty e^{-\lambda x}(1-e^{-x})^{\alpha-1} x = \frac{\Gamma(\lambda)\Gamma(\alpha)}{\Gamma(\lambda+\alpha)}.$$

Thank you for your attention!

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