# A local limit theorem for an oscillating random walk on $\mathbb{Z}$

joint work with Marc Peigné & Tran Duy Vo

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esaip
École d'Ingénieurs La Salle

LAREMA

Osillating random walk

Recurrence

Transience

Different subcases

Local limit theorem

Switching subprocess

Decomposition of trajectories

Renewal theorem

Sketch of the proof

$$X_{n+1} = X_n + \underbrace{\xi_{n+1}}_{\sim \mu,\ \text{iid}} \cdot \mathbb{1}[X_n < 0] + \underbrace{\xi'_{n+1}}_{\sim \mu',\ \text{iid}} \cdot \mathbb{1}[X_n > 0] + \underbrace{\eta_{n+1}}_{\sim \mu_0,\ \text{iid}} \cdot \mathbb{1}[X_n = 0]$$

$\mu' = \mu_0 = \mu \longrightarrow$ ordinary RW on $\mathbb{Z}$: $X_n = \xi_1 + \xi_2 + \dots + \xi_n$

$\mu' = \mu_0 = \check{\mu} \longrightarrow$ reflected RW on $\mathbb{N}$: $X_n = |X_{n-1} + \xi_n|$

polynomial or exponential moments
$\mu$ and $\mu'$ strongly aperiodic on $\mathbb{Z}$

Kemperman (1974)
$\hookrightarrow$ recurrent + transient

$$S_n = \xi_1 + \xi_2 + \dots + \xi_n$$

and $\quad S'_n = \xi'_1 + \xi'_2 + \dots + \xi'_n$

$$X_{n+1} = X_n + \underbrace{\xi_{n+1}}_{\mu,\ \text{iid}} \cdot \mathbb{1}[X_n < 0] + \underbrace{\xi'_{n+1}}_{\mu',\ \text{iid}} \cdot \mathbb{1}[X_n > 0] + \underbrace{\eta_{n+1}}_{\mu_0,\ \text{iid}} \cdot \mathbb{1}[X_n = 0]$$

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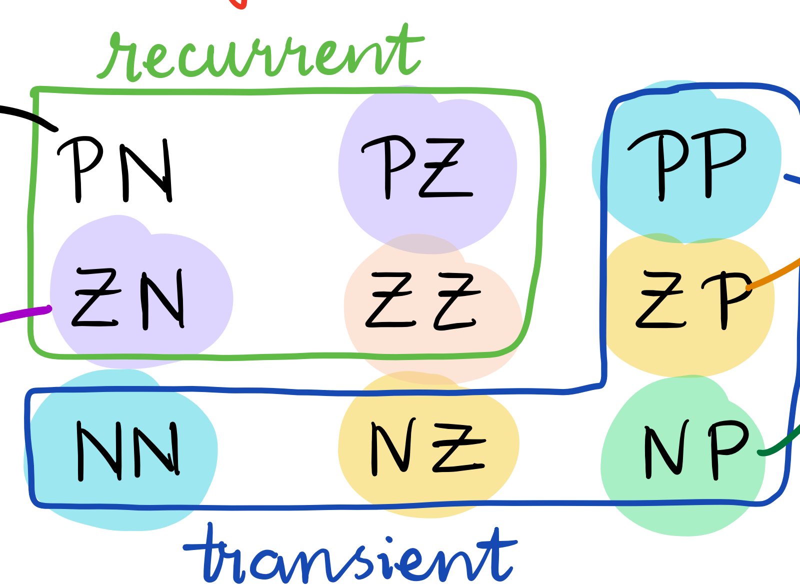
polynomial or exponential moments
$\mu$ and $\mu'$ strongly aperiodic on $\mathbb{Z}$

Kemperman (1974)
↳ recurrent + transient

$\forall x, y \in \mathbb{Z}$

(positive recurrent
$\mathbb{P}_x(X_n = y) \longrightarrow \nu(y)$




$\mathbb{P}_x(X_n = y) \sim \dfrac{c_{x,y}}{n^{3/2}}$

$\mathbb{P}_x(X_n = y) \sim \dfrac{c_{x,y}}{n^{3/2}} \max(\rho, \rho')^n$

$\mathbb{P}_x(X_n = y) \sim \dfrac{c_y}{\sqrt{n}}$

$\mathbb{P}_x(X_n = y) \sim \dfrac{c_{x,y}}{\sqrt{n}} \max(\rho, \rho')^n$

$c_{x,y}\, \rho_*^n$

$\dfrac{c_{x,y}}{n^{3/2}} \max(\rho, \rho')^n$

$$\rho := \lim_n \mathbb{P}_0(S_n = 0)^{1/n} = \inf_{t \in \mathbb{R}} \mathbb{E}[e^{t\xi}] =: L(\lambda)$$

where $S_n = \xi_1 + \xi_2 + \dots + \xi_n$

$\rho' := \dots$ and $S'_n = \xi'_1 + \xi'_2 + \dots + \xi'_n$

# Spitzer & Feller

Centered case $\mathbb{E}\xi = 0$

$$\mathbb{P}(\tau^s(-x) > n) \sim$$

$$\mathbb{P}(\tau^s(-x) > n, -x + S_n = -y) \sim$$

$$\mathbb{P}(\tau^s(-x) = n) \sim$$

$$\mathbb{P}(\tau^s(-x) = n, -x + S_n = y) \sim$$

$$\precsim$$

$$\tau^S(x) = \inf\{n \geqslant 1 : x + S_n \geqslant 0\}$$

$$\tau^S_{*+} = \inf\{k \geqslant 1, S_k > 0\}$$

$$\tau^S_- = \{ \cdots S_k \leqslant 0 \}$$

$$S_{\tau_{*+}} \longrightarrow M_{*+} \longrightarrow \mathcal{U}_{*+}$$

$$\mathcal{U}_{*+} := \sum_n (M_{*+})^{*n}$$

$$V_{*+}(x) = \mathcal{U}_{*+}[0; x]$$

$$S_{\tau_-} \longrightarrow M_- \quad \ldots$$

# Spitzer & Feller

Centered case $\mathbb{E}\xi = 0$, $\mathbb{E}(\xi^2) < +\infty$, aperiodicity : $\exists\, c > 0$ such that $\forall x, y \geqslant 1$ :

$$\mathbb{P}(\tau^S(-x) > n) \sim 2c \frac{V_{*+}(x)}{\sqrt{n}} \quad \text{and} \preccurlyeq \frac{1+x}{\sqrt{n}}$$

$$\mathbb{P}(\tau^S(-x) > n, -x + S_n = -y) \sim \frac{1}{\sigma\sqrt{2\pi}} \frac{V_{*+}(x) V_-(y)}{n^{3/2}} \preccurlyeq \frac{(1+x)(1+y)}{n^{3/2}}$$

$$\mathbb{P}(\tau^S(-x) = n) \sim \frac{c . V_{*+}(x)}{n^{3/2}} \quad \text{and} \preccurlyeq \frac{1+x}{n^{3/2}}$$

$$\mathbb{P}(\tau^S(-x) = n, -x + S_n = y) \sim \frac{1}{\sigma\sqrt{2\pi}} \cdot \frac{V_{*+}(x)}{n^{3/2}} \sum_{w \geqslant 1} V_-(w) . \mu(w+y)$$

$$\preccurlyeq \frac{1+x}{n^{3/2}} \sum_{z > y} z . \mu(z)$$

$\tau^S(x) = \inf\{n \geqslant 1 : x + S_n \geqslant 0\}$

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$V_{*+}(x) = \mathcal{U}_{*+}[0; x]$

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$$V_{*+}(x) = \mathcal{U}_{*+}[0; x]$$

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**Non centered case** $\mathbb{E}\xi \neq 0$, exponential moments + aperiodicity

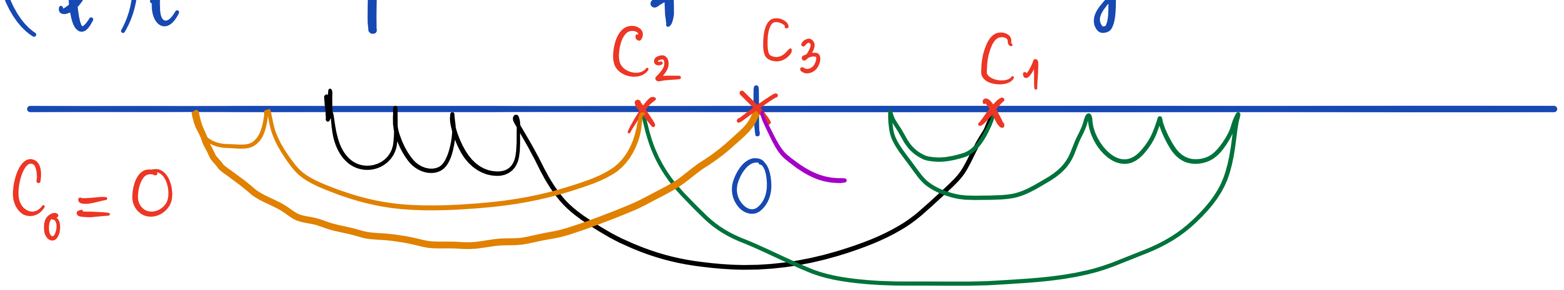
$\exists \lambda \neq 0$, $\forall x \leqslant -1$, $\forall y \geqslant 0$

$$\mathbb{P}(\tau^s(x) = n,\ x + S_n = y) \preccurlyeq \frac{\rho^n}{n^{3/2}}(1+|x|)\,e^{\lambda x} \sum_{z > y} z\,.\,e^{\lambda(z-y)}\,\mu(z)$$

$$\sim \frac{\rho^n}{\mathring{\sigma}\sqrt{2\pi}}\,\frac{\mathring{V}_{*+}(|x|)}{n^{3/2}}\,e^{\lambda(x-y)} \sum_{w \geqslant 1} \mathring{V}_-(w)\,.\,\mathring{\mu}(w+y)$$

$$\mathring{\mu}(z) := \frac{e^{\lambda z}}{\rho}\,\mu(z) \longrightarrow \mathring{\sigma} \quad \mathring{V}_{*+} \quad \mathring{V}_-$$

$(C_\ell)_\ell$ sequence of "switching times"




. time-homogeneous Markov chain on $\mathbb{Z}$

. transition kernel $Q(x,y) := \mathbb{E}_x[C_1 < +\infty, X_{C_1} = y]$

$$= \sum_{k \geq 1} \underbrace{\mathbb{E}_x[C_1 = k, X_k = y]}_{Q_k(x,y)}$$

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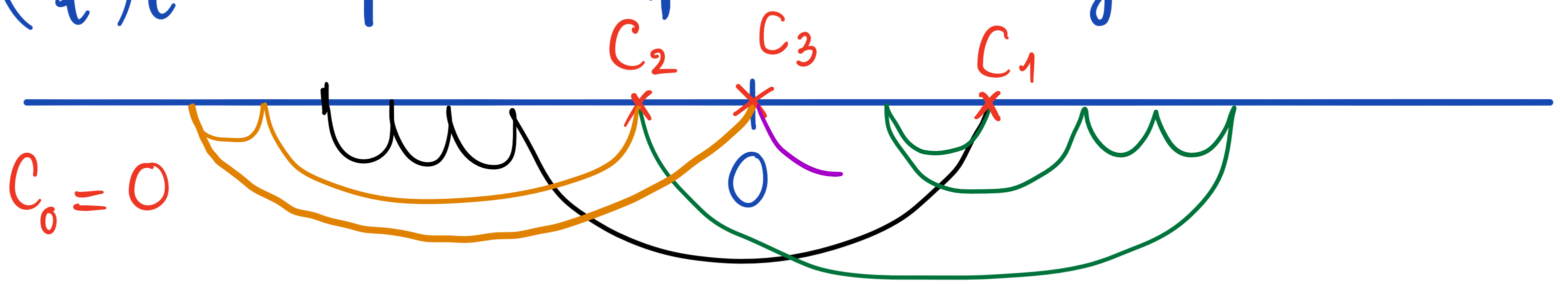


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Renewal theorem. Doney (1997)

If $(\tau_k)_k$ iid $\mathbb{N}^*$-RVs such that

1) $P(\tau_k > n) \sim \frac{c}{n^\beta}$ for $0 < \beta < 1$
$C, c > 0$

2) $P(\tau_k = n) \leq \frac{C}{n^{\beta+1}}$

3) $\tau_k$ are aperiodic

Then $T_k \longrightarrow \frac{1}{k^{1-\beta}} \cdot \frac{1}{\pi} \sin(\pi\beta)$

where $T_k := \sum_{\ell \geq 1} P(\tau_1 + \dots + \tau_\ell = k)$

$\forall x \in \mathbb{Z}, \forall y > 0$

$$\mathbb{P}_x(X_n = y) = \sum_{\ell \geq 0} \mathbb{P}_x(X_n = y, C_\ell \leq n < C_{\ell+1})$$

$$= \sum_{\ell \geq 0} \sum_{k=0}^{n} \sum_{z \geq 1} \underbrace{\mathbb{P}_x(C_\ell = k, X_k = z)}_{Q_k^{(\ell)}(x,z)} \cdot \underbrace{\mathbb{P}_z(S'_1 \geq 1, \dots, S'_{n-k-1} \geq 1, S'_{n-k} = y)}_{V_{n-k,y}(z) := P(\tau^{S'}(z) > n-k,\ z + S'_{n-k} = y)}$$

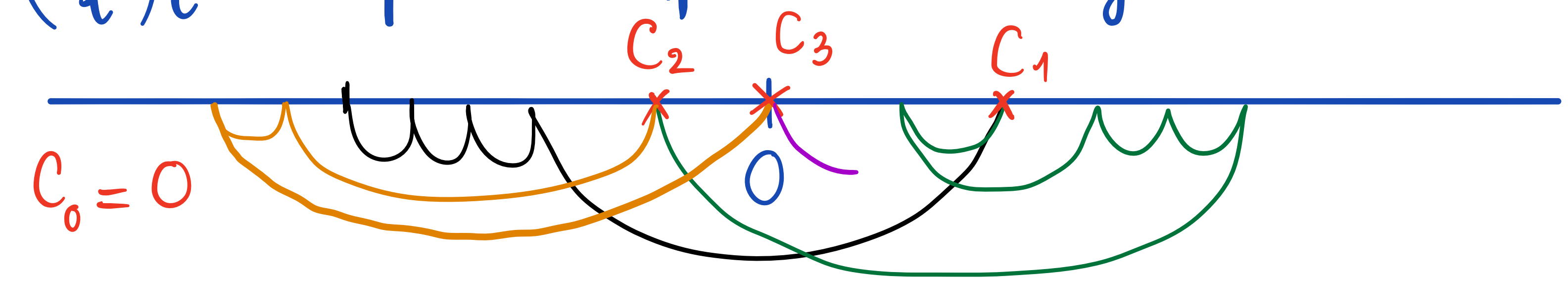
fluctuations of ordinary RW

$$= \sum_{k=0}^{n} \Big( \underbrace{\sum_{\ell \geq 0} Q_k^{(\ell)}}_{T_k} \Big) V_{n-k,y}(x)$$

$$= \sum_{k=0}^{n} \left( T_k V_{n-k,y} \right)(x)$$

$$T_k(x,y) = \sum_{\ell \geq 0} \mathbb{P}_x(C_\ell = k, X_k = y) \longrightarrow C_\ell = \sum_{0}^{\ell-1} (C_{i+1} - C_i)$$

$(C_\ell)_\ell$ sequence of "switching times"

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functional version

Renewal theorem

non iid RVs

## Renewal theorem. Doney (1997)

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## Gouëzel's theorem (2011) for an aperiodic renewal sequence of operators

$(R_n)_n$

- $R_n$ acts on $(\mathcal{B}, |\cdot|_{\mathcal{B}})$ Banach space
- $\sum_n \|R_n\|_{\mathcal{B}} < +\infty$
- $R(z) = \sum_n z^n R_n$, $z \in \overline{\mathbb{D}}$

R1 $R(1)$: simple eigenvalue $1$

R2 spect. rad. $(R(z)) < 1$ $\forall z \in \overline{\mathbb{D}} \setminus \{1\}$

R3 $\Pi_R R_n \Pi_R = \underbrace{r_n}_{\geq 0} R_n$ $\forall n \geq 1$

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**Theorem 1 : Behavior of** $T_n := \sum_{\ell \geq 1} R_n^{(\ell)}$, where $R_n^{(\ell)} := \sum_{j_1 + \dots + j_\ell = n} R_{j_1} \cdots R_{j_\ell}$

$(R_n)_n$ aperiodic renewal sequence of operators dans $(\mathcal{B}, |.|_\mathcal{B})$.

$\exists c > 0$ s.t. $\quad \|R_n\|_\mathcal{B} \leq \frac{C}{n^{1+\beta}}$ R4 $\quad$ and $\sum_{j>n} r_j \sim \frac{c}{n^\beta}$ R5

Then $n^{1-\beta} T_n \xrightarrow[n \to +\infty]{} \frac{c}{\pi} \sin(\pi\beta) . \Pi_R$ in $(\mathcal{L}(\mathcal{B}), \|.\|_\mathcal{B})$

## Sketch of the proof in the recurrent case $\mathbb{E}(\xi^{3+\delta}) < +\infty$.

PN / PZ and ZZ

$$\mathbb{P}_x(X_n = y) = \sum_{k=0}^{n} \left(T_k \mathbf{V}_{n-k,y}\right)(x) = \sum_{k=0}^{n} \left(\sum_{\ell} Q_k^{(\ell)}\right) \mathbf{V}_{n-k,y}(x)$$

$$Q_k^{(\ell)}(x,y) = \mathbb{P}_x(C_\ell = n, X_n = y)$$

↳ distribution of $C_\ell$

$$C_{\ell+1} = \begin{cases} \tau^{S}(X_{C_\ell}), & X_{C_\ell} < 0 \\ \tau^{S'}(X_{C_\ell}), & X_{C_\ell} > 0 \end{cases}$$

↳ classical result

↳ choose Banach space

$$\mathcal{B}_\psi := \left\{ f : \mathbb{Z} \to \mathbb{C},\ |f|_{\mathcal{B}_\psi} := \sup_x \frac{|f(x)|}{|\psi(x)|} < +\infty \right\}, \text{ where } \psi(x) = 1 + |x|^{1+\delta}$$

$(\delta > 0)$

## Sketch of the proof in the recurrent case $\mathbb{E}(\xi^{3+\delta}) < +\infty$.

PN / PZ and ZZ $\quad \mathbb{P}_x(X_n = y) = \sum_{k=0}^{n} \left(T_k \mathbf{V}_{n-k,y}\right)(x) = \sum_{k=0}^{n} \left(\sum_{\ell} Q_k^{(\ell)}\right) \mathbf{V}_{n-k,y}(x)$

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classical result

1) $Q$ acts on $\mathcal{B}_\psi$, markovian + compact operator on $\mathcal{B}_\psi$

$\rho_\psi = 1$ and unique simple dominant eigenvalue $1$

$Q = \Pi_Q + Q'$ $\quad \begin{cases} \Pi_Q(\varphi) = \nu_Q(\varphi) \cdot \mathbb{1}, & \nu_Q \text{ unique invariant measure} \\ \text{spect.}(Q') < 1 & \\ \Pi_Q \cdot Q' = Q' \cdot \Pi_Q = 0 & \end{cases}$

$Q = \sum_n Q_n$

choose Banach space

2) Prove that $(Q_n)_n$ aperiodic renewal sequence in $\mathcal{B}_\psi$ satisfying R4 and R5

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choose Banach space

1) $Q$ acts on $\mathcal{B}_\psi$, markovian + compact operator on $\mathcal{B}_\psi$

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$Q = \sum_n Q_n$

2) Prove that $(Q_n)_n$ aperiodic renewal sequence in $\mathcal{B}_\psi$ satisfying R4 and R5

3) Applying Theorem 4.1 (Gouëzel) with $\beta = \frac{1}{2}$

$$\sqrt{n} \cdot T_n \xrightarrow[n \to +\infty]{} c^{-1} \cdot \Pi_Q \quad \text{in } \left(\mathcal{L}(\mathcal{B}), \|\cdot\|_{\mathcal{B}}\right)$$

$$\sqrt{n}\, T_n(x,y) \longrightarrow c^{-1} \cdot \nu_Q(y)$$

4) Using a functional version of Iglehart's lemma to obtain the asymptotic behavior.

## Sketch of the proof in the transient case $\mathbb{Z}P$ $\quad \mathbb{E}(\xi^{3+\delta}) < +\infty$.

$Q$ is submarkovian + compact on $\mathcal{B}_\psi$, $\rho_\psi < 1$

$$Q \xrightarrow[\text{Banach space } \mathcal{B}_\psi]{\text{Doob's } H \text{ transform}} {}^H Q \text{ is Markovian + compact + } {}^H\rho_\psi = 1$$

$({}^H Q_n)_n$ aperiodic renewal sequence of operators in $\mathcal{B}_\psi$

$$\lim_n n^{3/2}\, \mathbb{P}_x(X_n = y) = \lim_n n^{3/2} \sum_{k=0}^{n} \sum_{\ell} \left(Q_k^{(\ell)} V_{n-k,y}\right)(x)$$

$$= \lim_n n^{3/2}\, H(x) \sum_{\ell=0}^{+\infty} \rho_\psi^{\ell} \left[ \sum_{k=0}^{n} {}^H Q_k^{(\ell)} \left(V_{n-k,y} | H\right)(x) \right]$$

$$= H(x). \sum_{\ell \geq 0} \rho_\psi^{\ell} \left[ \sum_{k \geq 0} {}^H Q_k^{(\ell)} \left(V_y | H\right)(x) + \sum_{k \geq 0} {}^H \mathcal{E}_\ell \left(V_{n,y} | H\right)(x) \right]$$

## Sketch of the proof in the transient case ZP $\mathbb{E}(\xi^{3+\delta}) < +\infty$.

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$$= \lim_n n^{3/2} H(x) \sum_{\ell=0}^{+\infty} \rho_\psi^\ell \left[\sum_{k=0}^{n} {}^{H}Q_k^{(\ell)} \left(V_{n-k,y} | H\right)(x)\right]$$

$$= H(x). \sum_{\ell \geq 0} \rho_\psi^\ell \left[\sum_{k\geq 0} {}^{H}Q_k^{(\ell)} \left(V_y | H\right)(x) + \sum_{k\geq 0} {}^{H}\mathcal{E}_\ell \left(V_{n,y} | H\right)(x)\right]$$

## Theorem 2: Behavior of $R_n^{(\ell)}$

If $\exists \beta > 0$ s.t. $n^{1+\beta} R_n \longrightarrow \mathcal{E}$ bounded operator in $(\mathcal{L}(\mathcal{B}), \|\cdot\|_\mathcal{B})$

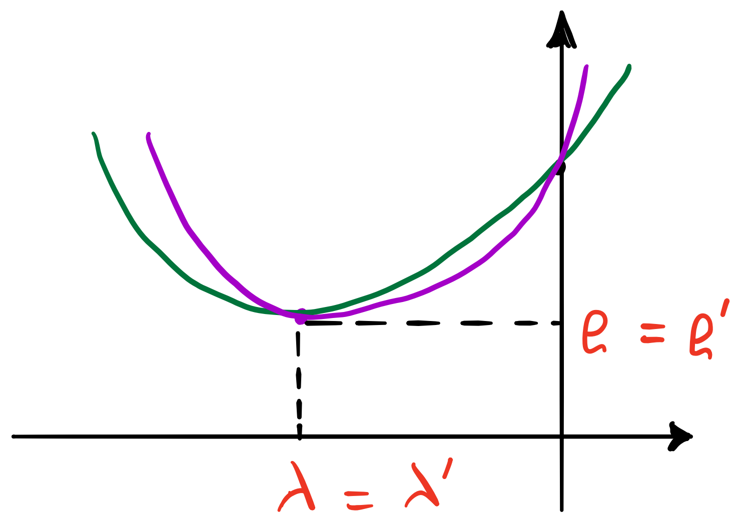
Then
- $\exists c > 0, \forall n, \ell \geq 1 : \quad \|R_n^{(\ell)}\|_\mathcal{B} \leq c . \dfrac{\ell^2}{n^{1+\beta}}$
- $\forall \ell \geq 1 \quad n^{1+\beta} . R_n^{(\ell)} \longrightarrow \mathcal{E}_\ell := \sum_{i=0}^{\ell-1} R^{(i)} \mathcal{E} R^{(\ell-i-1)}$

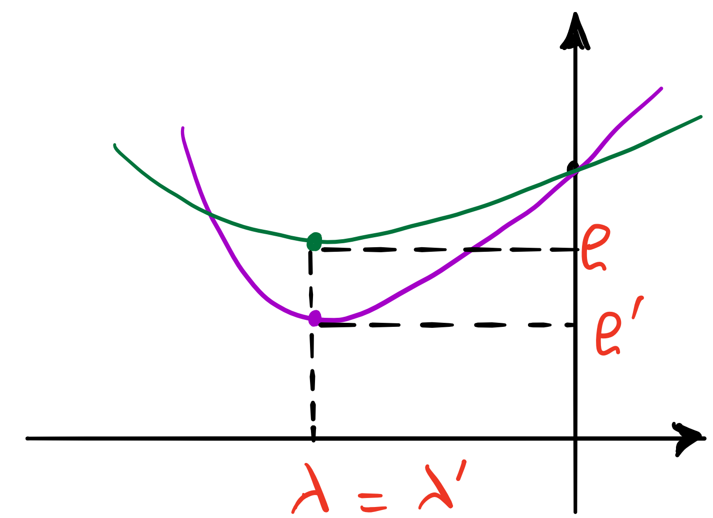
# A few remarks in the other transient subcases NP and PP

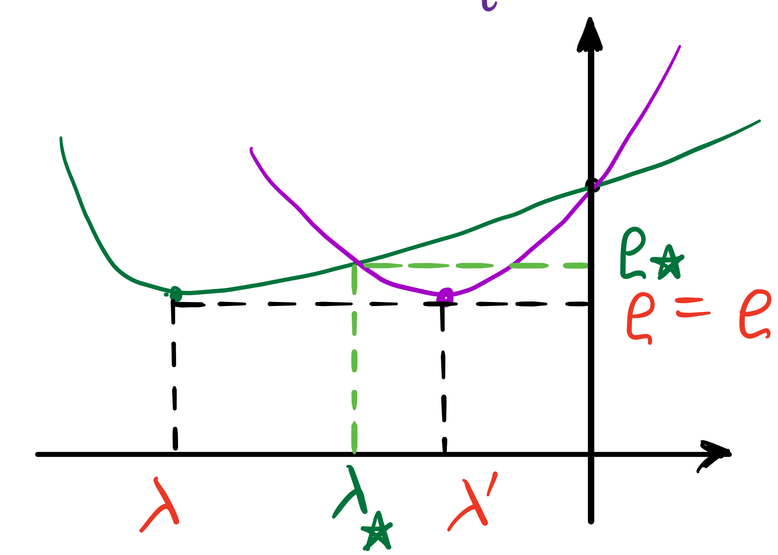
**NP:** Extraction of $\max(\rho,\rho')^n \longrightarrow$ Doob's H $\longrightarrow$ Theorem 2. (like ZP).

**PP:** Relative positions of the graphs of $L$ and $L'$

$$\rho = \min_t L(t) = L(\lambda)$$
$$\rho' = \min_t L'(t) = L'(\lambda')$$




$$\mathbb{P}_x(X_n = y) \sim \frac{1}{\sqrt{n}} \max(\rho,\rho')^n \qquad \sim \frac{1}{n^{3/2}} \max(\rho,\rho')^n \qquad \sim \rho_\star^n \, e^{\lambda_\star (x-y)} \nu_\star(y)$$

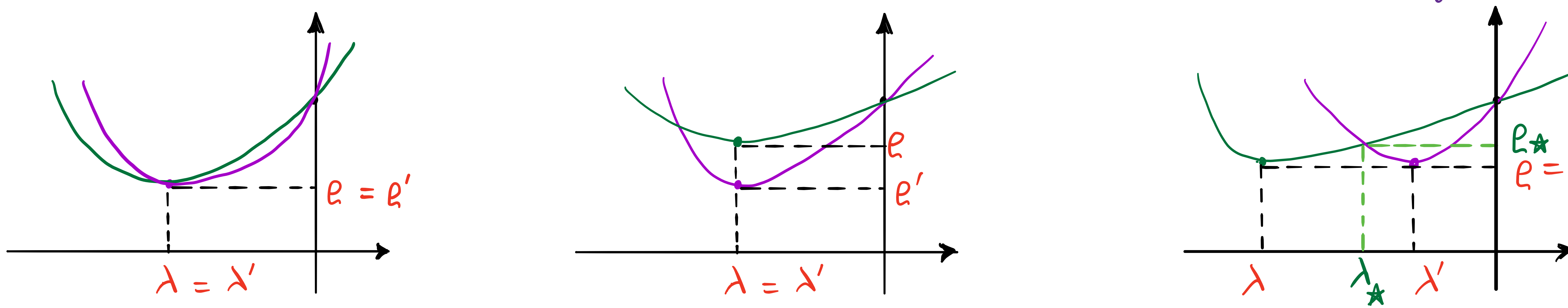
# A few remarks in the other transient subcases NP and PP

NP: Extraction of $\max(\rho,\rho')^n \longrightarrow$ Doob's H $\longrightarrow$ Theorem 2. (like ZP).

PP: Relative positions of the graphs of L and L'

$\rho = \min_t L(t) = L(\lambda)$

$\rho' = \min_t L'(t) = L'(\lambda')$




$\mathbb{P}_x(X_n = y) \sim \frac{1}{\sqrt{n}} \max(\rho,\rho')^n \qquad \sim \frac{1}{n^{3/2}} \max(\rho,\rho')^n \qquad \sim \rho_\star^n \, e^{\lambda_\star(x-y)} \nu_\star(y)$

**Doney** "One sided local large deviation and renewal theorems in the case of infinite mean". (1997)

**Gouëzel** "Correlation asymptotics from large deviations in dynamical system with infinite measure". (2011)

**Iglehart** "Random walks with negative drift conditioned to stay positive" (1974)

**Kemperman** "The oscillating random walk". (1974)

Thank you for your attention.

The end.