# Quenched critical percolation on Galton–Watson trees

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**Joint works with Quirin Vogel and Tanguy Lions**

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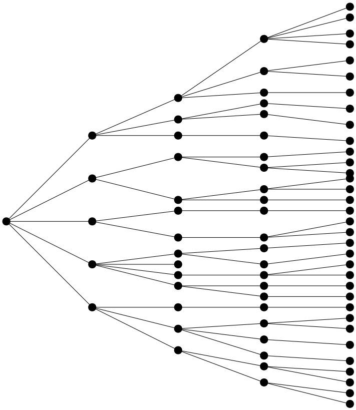
# The model

Let $T$ be a supercritical Galton–Watson tree with no leaves, with offspring law $\xi$ satisfying $\mathbb{E}[\xi^\alpha] < \infty$ for some $\alpha > 1$. Let $\mu > 1$ be the mean number of offspring, and $\mathbf{o}$ the root.



...

(no leaves means almost sure survival - convenient)

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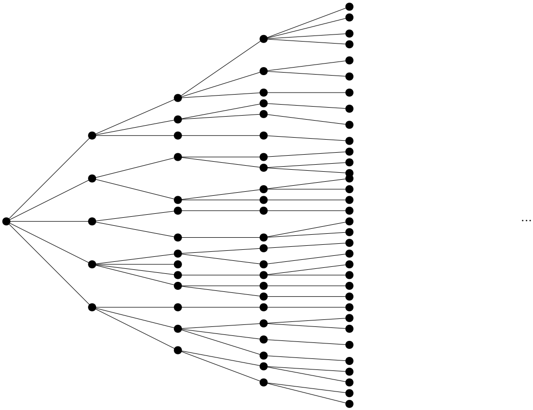

**Important fact:** let $Z_n$ be the number of individuals at generation $n$. Then $\exists$ r.v. $W$, supported on $(0, \infty)$, such that, a.s., $\frac{Z_n}{\mu^n} \to W$.

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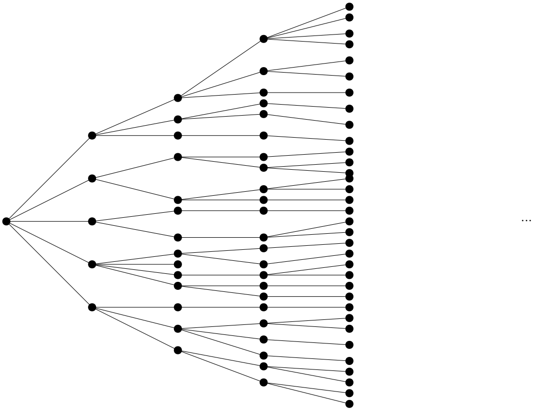

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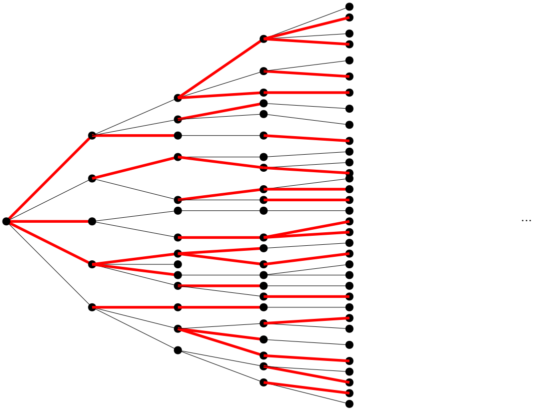

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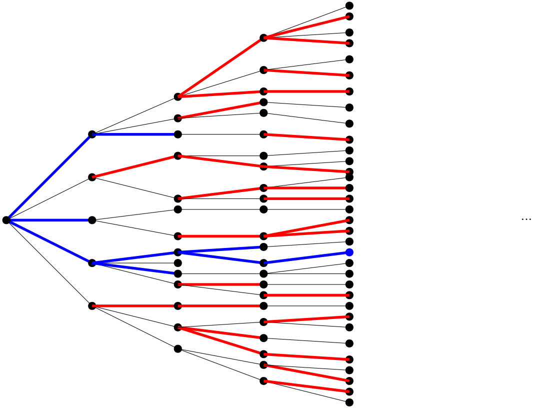


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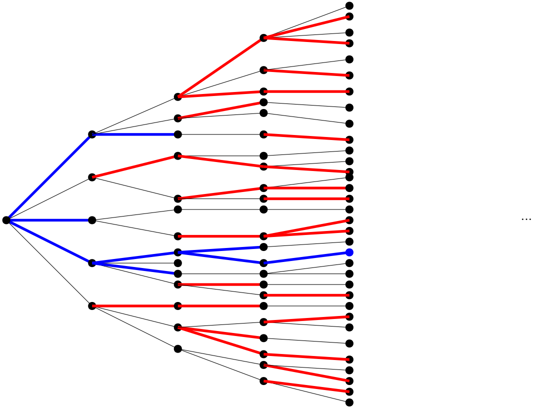
# Percolation on $T$



As usual, we define:

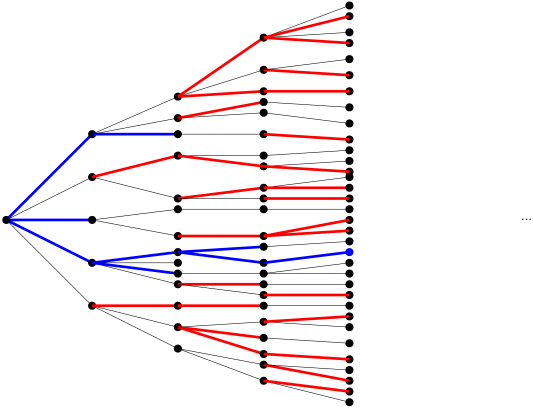
$$p_c = \inf\{p > 0 : \mathbb{P}\left(\mathbf{o} \overset{p}{\longleftrightarrow} \infty\right) > 0\}.$$

# What is $p_c$?

Let $C$ denote the root cluster (the blue structure).

**Observation: $C$ has the law of a Galton-Watson tree.**

Offspring law: first sample $N \sim \xi$, then take Binomial$(N, p)$.




Hence: $\mathbb{P}\left(\mathbf{o} \overset{p}{\longleftrightarrow} \infty\right) > 0$ iff mean $> 1$, i.e. iff $\mathbb{E}[Np] > 1$.

Hence $p_c = 1/\mu$.

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Another point of view: for a given realisation of $T$, we can set

$$p_c(T) = \inf\left\{p > 0 : \mathbb{P}_T\left(\mathbf{o} \overset{p}{\longleftrightarrow} \infty\right) > 0\right\},$$

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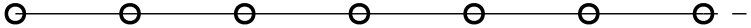
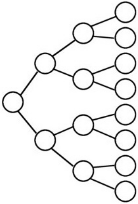
**Question:** is it true that $p_c(T) = 1/\mu$ for almost every realisation of $T$?

# Example of a random tree where there is a difference

Consider the random tree $\tilde{T}$ which is equal to $T_1$, the 1-regular tree, with prob $1/2$, and $T_2$, the 2-regular tree, with prob $1/2$.





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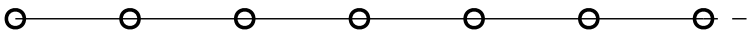
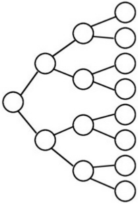


$p_c(T_1) = 1$



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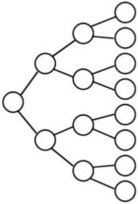

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Overall $p_c = 1/2$ since for $p < 1/2$,

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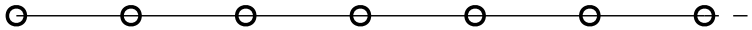

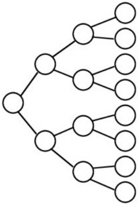

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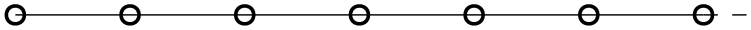

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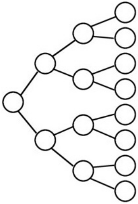

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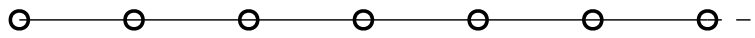

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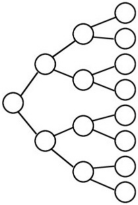

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For our supercritical GW: $p_c = 1/\mu$, almost surely (Lyons 1990).

# Quenched vs annealed results: notation

$T_n = n^{th}$ generation of $T$, $C =$ cluster of $\mathbf{o}$, $Y_n = |C \cap T_n|$,
$C_{\geq n} = C$ conditioned to have size $n$, $W = \lim \frac{Z_n}{\mu^n}$

$\mathbf{P} =$ law of $T$
$\mathbb{P}_T =$ law of percolation on $T$, given $T$
$\mathbb{P} = \mathbf{P} \times \mathbb{P}_T$, annealed law

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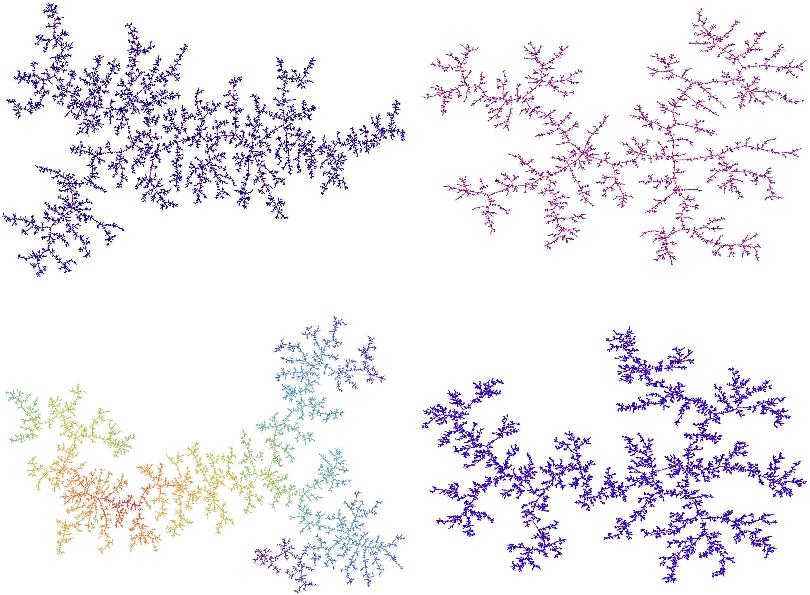
# The CRT



Pictures by Igor Kortchemski and Laurent Ménard.

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| $\mathbb{P}\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right) \sim cn^{-1}$ | $\mathbb{P}_T\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right) \sim W \cdot cn^{-1}$ a.s. *+ |
| $\mathbb{P}(\lvert C\rvert \geq n) \sim c' n^{-1/2}$ | $\mathbb{P}_T(\lvert C\rvert \geq n) \sim W \cdot c' n^{-1/2}$ a.s. |
| Given $Y_n > 0$: $n^{-1} Y_n \overset{(d)}{\to} Y$ | $n^{-1} Y_n^T \overset{(d)}{\to} Y$ a.s. * |
| and $\left(n^{-1} Y_{n(1+t)}\right)_{t \geq 0} \overset{(d)}{\to} (Y_t)_{t \geq 0}$ | $\left(n^{-1} Y^T_{n(1+t)}\right)_{t \geq 0} \overset{(d)}{\to} (Y_t)_{t \geq 0}$ a.s. |
| Given $Y_\infty > 0$: $n^{-1} Y_n \overset{(d)}{\to} \hat{Y}$ | $n^{-1/2} Y_n^T \overset{(d)}{\to} \hat{Y}$ a.s. * |
| $(C_{\geq n}, n^{-1/2} d_n, \frac{1}{n}\mu_n) \underset{GHP}{\overset{(d)}{\to}} CRT$ | $(C^T_{\geq n}, n^{-1/2} d_n, \frac{1}{n}\mu_n) \underset{GHP}{\overset{(d)}{\to}} CRT$ a.s. |

*proved by Michelen (2019) under higher moment assumptions.

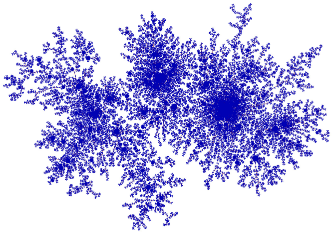
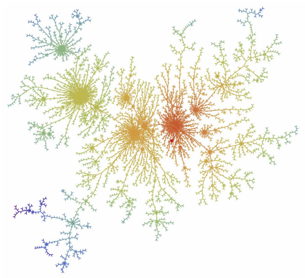
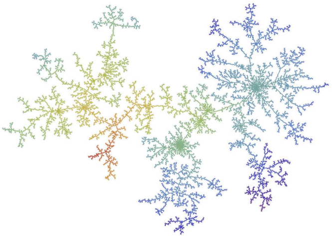
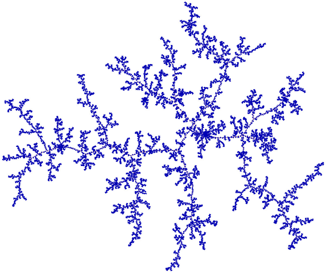
+also obtained by Berger-Ayuso Ventura (2024).

# Quenched vs annealed results - stable analogues

Stable tails on offspring law: $\xi(x,\infty) \sim cx^{-\alpha}$, where $\alpha \in (1,2)$.
$T_n = n^{th}$ generation of $T$, $C =$ cluster of $\mathbf{o}$, $Y_n = |C \cap T_n|$, $C_{\geq n} = C$ conditioned to have size $n$, $W = \lim \frac{Z_n}{\mu^n}$. [AV24], [AL25].

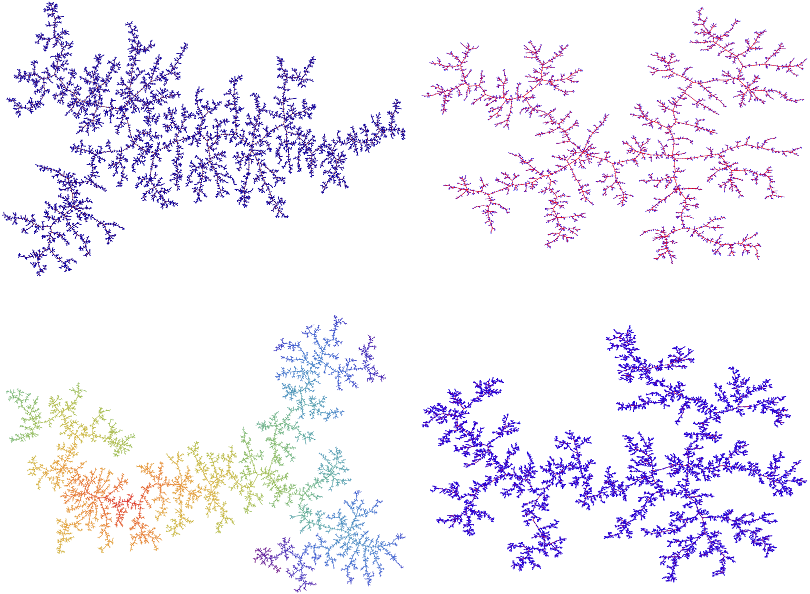
| **Annealed** | **Quenched** |
|---|---|
| $p_c = 1/\mu$ | $p_c(T) = 1/\mu$ a.s. |
| $\mathbb{P}\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right) \sim cn^{-\frac{1}{\alpha-1}}$ | $\mathbb{P}_T\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right) \sim W \cdot cn^{-\frac{1}{\alpha-1}}$ |
| $\mathbb{P}(\|C\| \geq n) \sim c' n^{-1/\alpha}$ | $\mathbb{P}_T(\|C\| \geq n) \sim W \cdot c' n^{-1/\alpha}$ |
| Given $Y_n > 0$: $n^{-\frac{1}{\alpha-1}} Y_n \overset{(d)}{\to} Y$ | $n^{-\frac{1}{\alpha-1}} Y_n^T \overset{(d)}{\to} Y$ a.s. |
| $\left(n^{-\frac{1}{\alpha-1}} Y_{n(1+t)}\right)_{t\geq 0} \overset{(d)}{\to} (Y_t)_{t\geq 0}$ | $\left(n^{-\frac{1}{\alpha-1}} Y^T_{n(1+t)}\right)_{t\geq 0} \overset{(d)}{\to} (Y_t)_{t\geq 0}$ a.s. |
| Given $Y_\infty > 0$: $n^{-\frac{1}{\alpha-1}} Y_n \overset{(d)}{\to} \hat{Y}$ | $n^{-\frac{1}{\alpha-1}} Y_n^T \overset{(d)}{\to} \hat{Y}$ |
| $(C_{\geq n}, n^{-(1-1/\alpha)} d_n, \frac{1}{n}\mu_n) \underset{\text{GHP}}{\overset{(d)}{\to}} \mathcal{T}_\alpha$ | $(C^T_{\geq n}, n^{-(1-1/\alpha)} d_n, \frac{1}{n}\mu_n) \underset{\text{GHP}}{\overset{(d)}{\to}} \mathcal{T}_\alpha$ a.s. |

# Stable trees

Pictures by Igor Kortchemski.

# The CRT



Pictures by Igor Kortchemski and Laurent Ménard.

# Quenched vs annealed results - finite variance

$T_n = n^{th}$ generation of $T$, $C$ = cluster of $\mathbf{o}$, $Y_n = |C \cap T_n|$,
$C_{\geq n} = C$ conditioned to have size $n$, $W = \lim \frac{Z_n}{\mu^n}$. [AV24], [AL25].

| **Annealed** | **Quenched** |
|---|---|
| $p_c = 1/\mu$ | $p_c(T) = 1/\mu$ a.s. |
| $\mathbb{P}\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right) \sim cn^{-1}$ | $\mathbb{P}_T\left(\mathbf{o} \overset{p_c}{\longleftrightarrow} T_n\right) \sim W \cdot cn^{-1}$ a.s. *+ |
| $\mathbb{P}(\lvert C\rvert \geq n) \sim c' n^{-1/2}$ | $\mathbb{P}_T(\lvert C\rvert \geq n) \sim W \cdot c' n^{-1/2}$ a.s. |
| Given $Y_n > 0$: $n^{-1} Y_n \overset{(d)}{\to} Y$ | $n^{-1} Y_n^T \overset{(d)}{\to} Y$ a.s. * |
| and $\left(n^{-1} Y_{n(1+t)}\right)_{t \geq 0} \overset{(d)}{\to} (Y_t)_{t \geq 0}$ | $\left(n^{-1} Y^T_{n(1+t)}\right)_{t \geq 0} \overset{(d)}{\to} (Y_t)_{t \geq 0}$ a.s. |
| Given $Y_\infty > 0$: $n^{-1} Y_n \overset{(d)}{\to} \hat{Y}$ | $n^{-1/2} Y_n^T \overset{(d)}{\to} \hat{Y}$ a.s. * |
| $(C_{\geq n}, n^{-1/2} d_n, \frac{1}{n}\mu_n) \underset{\text{GHP}}{\overset{(d)}{\to}} CRT$ | $(C^T_{\geq n}, n^{-1/2} d_n, \frac{1}{n}\mu_n) \underset{\text{GHP}}{\overset{(d)}{\to}} CRT$ a.s. |

*proved by Michelen (2019) under higher moment assumptions.
+also obtained by Berger-Ayuso Ventura (2024).

# Extension to critical percolation on hyperbolic random planar maps??

$C$ = cluster of $\mathbf{o}$, $C_{\geq n} = C$ conditioned to have size $n$,

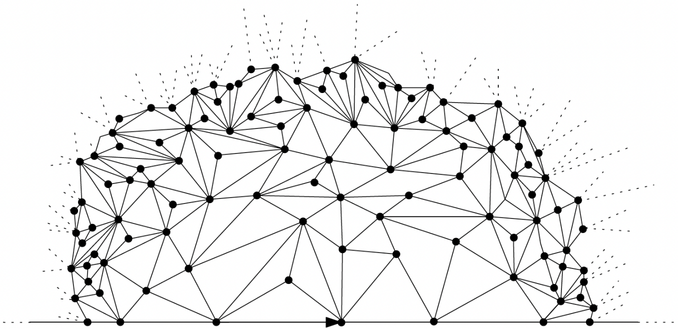


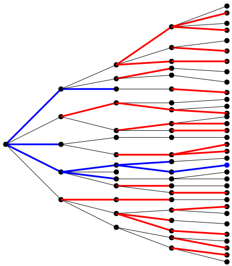
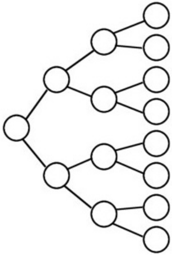
Image by Gourab Ray.

# Extension to critical percolation on hyperbolic random planar maps??

$C$ = cluster of $\mathbf{o}$, $C_{\geq n} = C$ conditioned to have size $n$.

| **Annealed** | **Quenched** |
| --- | --- |
| $p_c$ is explicit (Ray 2014) | |
| $\mathbb{P}(\text{Height}(C) \geq n) \sim cn^{-1}$ * | |
| $\mathbb{P}(\lvert C \rvert \geq n) \sim c' n^{-1/2}$ * | |
| $(C_{\geq n}, n^{-1/2} d_n, \frac{1}{n}\mu_n) \overset{(d)}{\underset{\text{GHP}}{\rightarrow}} CRT$ * | |

*A.-Croydon 2023.

Thank you!!