Martin boundaries and asymptotic behavior of supercritical branching random walks

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Model

Ingredients:

- **Graph** $G = (V, E)$ (infinite, locally finite, connected);
- Transition kernel on G, call it $P := (p(x, y))_{x,y\in V}$ (for this talk: **nearest neighbor**, i.e., $p(x, y) > 0 \Leftrightarrow x \sim y$);

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- **Offspring distribution** ν with mean $\mathbf{m} := \sum_{k \geq 0} k\nu(k)$ s.t.

$$\begin{cases} \mathbf{m} > 1 \quad \text{(survival wpp)} & (*) \\ \mathbb{E}(L \log(L)) < \infty, \quad L \sim \nu. & (**) \end{cases}$$

Branching Random Walks (BRWs)

Model for growth of population of bacteria

- Start with **1 individual** at a reference vertex o;
- Inductively: at each generation, each alive particle produces, **independently** a **random number of offspring**. This number is distributed according to ν;
- Each newly-born particle takes 1 step according to P, while old particles die.

For $n \geq 0$ let $\mathbf{B}_n \in \mathbb{N}^V$ be the vector s.t. for all $x \in V$

$$\mathbf{B}_n(x) := \# \text{ individuals alive at } x \text{ at time } n.$$

Branching Random Walks (BRWs)

Note: Known [Galton-Watson] that $\mathbf{m} > 1$ $(*)$ $\Rightarrow$ **indefinite survival** occurs with positive probability.

Note: Known [Benjamini-Peres + Gantert-Müller, Bertacchi-Zucca] that if $\mathbf{m} \leq \frac{1}{\rho_G}$ then BRW **transient** on G. (ρ_G = spectral radius of P.)

Note: $\exists$ random variable $W \geq 0$ s.t. almost surely $\lim_n \frac{\sum_{x \in V} \mathbf{B}_n(x)}{\mathbf{m}^n} = W$. By $L \log L$-condition $(**)$ $\Rightarrow \mathbb{E}W = 1$ [Kesten-Stigum].

Random Walks and Martin Boundaries I

Recap

(i) A function $h : V \to \mathbb{R}$ is harmonic if, for all $x \in V$,

$$h(x) = \sum_{y \in V} p(x,y)h(y) =: Ph(x);$$

(ii) Martin kernel: for all $x, y \in V$

$$K(x,y) := \frac{G(x,y)}{G(o,y)},$$

where $G(x,y) := \sum_{n \geq 0} p^{(n)}(x,y)$ is the Green function.

Random Walks and Martin Boundaries II

- The Martin boundary is a **topological** space (related to nonnegative harmonic functions).
- Its construction depends on the **Martin kernels** $K(x, \cdot)$, which depend on P $\Rightarrow$ by changing P we change the Martin boundary.
- A sequence of states (y_n) converges in the Martin topology

$$\Updownarrow$$

$$(K(x, y_n)) \text{ converges for all } x \in V.$$

Notation: if $(y_n) \to \xi \in \partial G$, then set $K(x, \xi) := \lim_n K(x, y_n)$ for all $x \in V$.

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Classical Convergence Theorem [Doob]. Let (X_n) be a **transient** RW on G governed by P, then $\exists$ **unique random variable** $X_\infty \in \partial G$ s.t. $\forall x \in V$,

$$\lim_n X_n = X_\infty \quad \mathbb{P}_x\text{-almost surely}$$

in the Martin topology. Moreover, for all Borel sets $A \subset \partial G$,

$$\mathbb{P}_x(X_\infty \in A) = \int_A K(x,\cdot)d\gamma(\cdot),$$

for some measure γ supported on ∂G.

Note: $K(o,y) = 1$ for all $y \in \widehat{G}$.

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$$\left(\frac{\mathbf{B}_n}{\mathbf{m}^n}\right) \text{ converges weakly to a } \underline{\text{random measure}}\ \mathbf{W} \text{ on } \partial G.$$

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$$\mathbb{E}_o \mathbf{W}(A) = P_o(X_\infty \in A), \quad \text{for all } A \text{ Borel set of } \partial G.$$

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Fix an infinite subgraph $U \subset G$, with 2 transition kernels:

$$P_U \quad \text{s.t.} \quad p_U(x,y) := \begin{cases} p(x,y) & \text{if } x,y \in U \\ 0 & \text{otherwise} \end{cases}$$

and

$$Q_U \quad \text{s.t.} \quad q_U(x,y) := \begin{cases} p_U(x,y) & \text{if } \sum_{w \in U} p_U(x,w) = 1 \\ \frac{p_U(x,y)}{\sum_{w \in U} p_U(x,w)} & \text{otherwise} \end{cases}$$

Note: Q_U is transition kernel of RW conditional on staying in U.

Survival vs Persistence

Survival of BRW in U: when U is visited infinitely often by particles of BRW (**They can enter and exit U!**)

Persistence of BRW in U: there is a particle born in U s.t. there is a line of descendants completely contained in U (**This one never exits U!**)

Interested in survival/persistence/convergence of trajectories of BRW started in U to some Borel set in ∂G.

Survival vs Persistence: results I

Spectral radius of RW governed by P: $\rho_G := \limsup_n \sqrt[n]{p^{(n)}(o,o)}$.

Let ρ_U and ϕ_U denote the spectral radii of P_U and Q_U respectively.

Theorem 1 [**Bertacchi** - **C.** - **Zucca**] Suppose U is **connected**.

- We have

$$\text{If } \forall\, \mathbf{m} > 1,\ \mathbb{P}_x\left((\mathbf{B}_n) \text{ persists in } U\right) > 0 \ \text{ then } \ \rho_U = \phi_U.$$

  If U is "**regular enough**" then also vice versa holds.

- Moreover,

$$P_x^P(X_\infty \in \partial U) > 0 \ \Rightarrow\ \forall\, \mathbf{m} > 1,\ \mathbb{P}_x\left((\mathbf{B}_n) \text{ persists in } U\right) > 0.$$

Survival vs Persistence: results I – comments

- In Thm 1, is it true that $\rho_U = \phi_U \quad \Rightarrow \quad \rho_U = \phi_U = \rho_G$?

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No.

There are examples where $\rho_G > \rho_U$. (Ex. Homogeneous tree attached to a singleton, with a suitable transition probability.)

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  There are examples where $\rho_G > \rho_U$. (Ex. Homogeneous tree attached to a singleton, with a suitable transition probability.)

- We say that U is "**regular enough**" when the *induced* Galton-Watson process has only *finitely* many types.

Survival vs Persistence: results II

If ρ_U and ϕ_U denote the spectral radii of P_U and Q_U respectively, let

$$\zeta := \frac{\rho_U}{\phi_U} \leq 1.$$

Theorem 2 [**Bertacchi - C. - Zucca**] Suppose U regular (*1 type GW*), infinite, then

- $\partial U(Q_U, 1)$ homeomorphic to $\partial U(P_U, \zeta)$.
- When $\mathbf{m} > 1$ and $L \log L$ condition holds, then **almost surely**

$$\left(\frac{\mathbf{B}_n}{(\mathbf{m}\zeta)^n}\right) \text{ converges weakly to a random measure } \mathbf{W}_\zeta \text{ on } \partial U(P_U, \zeta)$$

and $\mathbb{E}_x(\mathbf{W}_\zeta(A)) = P_x^{P_U}(Y_\infty \in A)$ for all A Borel set of $\partial U(P_U, \zeta)$, $Y_\infty = \lim_n Y_n$ and (Y_n) denotes a RW governed by P_U.

Survival vs Persistence: results II – comments

- Some meaning of Thm 2: when considering a **subgraph**

$$\Downarrow$$

renormalization factor might be different than $\mathbf{m}$.

We find:

(i) Where the process converges;

(ii) What the renormalization factor is.

Survival vs Persistence: results III

If ρ_U and ϕ_U denote the spectral radii of P_U and Q_U respectively, let

$$\zeta := \frac{\rho_U}{\phi_U} \leq 1.$$

Theorem 3 [**Bertacchi - C. - Zucca**] Suppose U connected and "regular enough". Then,

$$\mathbf{m} > \frac{1}{\zeta} \;\Rightarrow\; \mathbb{P}_x\left((\mathbf{B}_n) \text{ persists in } U\right) > 0.$$

Remark: the value $\frac{1}{\zeta}$ represents the equivalent of "1" for standard BRW!

Survival vs Persistence: Some proof ideas

Main point: understand the "asymptotic" properties of the process on U. Suppose U connected,

- Let (Y_n) denote a RW with kernel P. Then,

$$\rho_U = \limsup_n \left(p_U^{(n)}(x,x)\right)^{1/n} =$$

$$= \limsup_n \left[\mathbb{P}_x(Y_n = x \mid Y_1, \dots, Y_n \in U)\mathbb{P}(Y_1, \dots, Y_n \in U)\right]^{1/n}$$

$$= \phi_U \limsup_n (\mathbb{P}(\text{RW doesn't exit } U \text{ for } n \text{ steps}))^{1/n}.$$

Thus (when U is regular enough),

$$\zeta \approx \mathbb{P}(\text{RW takes one step in } U)$$

Survival vs Persistence: Some proof ideas

U connected, $\left(\mathbb{E}\left[\#\text{particles that at time } n \text{ are in } U\right]\right)^{1/n} \to \mathbf{m}\,\frac{\rho_U}{\phi_U}$.

- Persistence $\Rightarrow \mathbf{m}\,\frac{\rho_U}{\phi_U} \geq 1$. If this holds for all $\mathbf{m} > 1$, then $\frac{\rho_U}{\phi_U} = 1$.
- U "regular enough"

  $\Downarrow$

  If $\frac{\rho_U}{\phi_U} = 1 \;\; \Rightarrow \;\;$ persistence.

Examples and counterexamples I

Is it possible $P^P(X_\infty \in \partial U) = 0$ but $\mathbb{P}((\mathbf{B}_n)$ persists in $U) > 0$?

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YES!

Example: $G := T_3 \times T_{100}$, $U := o_1 \times T_{100}$ with P as SRW on each factor.

Here the range (easy computations)

$$\frac{1}{\zeta} < \mathbf{m} \leq \frac{1}{\rho_G}$$

is non-empty, hence by **Thm 3**, $\mathbb{P}((\mathbf{B}_n)$ persists in $U) > 0$ (and BRW transient on G).

However, known [Picardello-Woess] that $P^P(X_\infty \in \partial U) = 0$!

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Now: does "**large**" boundary imply survival?

NO!

(Counter)Example: Let G be the **homogeneous tree** $\mathbb{T}_d$.
We show that for *some BRW* (no details here) on $\mathbb{T}_d$, for any subgraph $B \subset \mathbb{T}_d$ there is a set $A_B \subset \mathbb{T}_d$ so that

$$\partial A_B = \partial \mathbb{T}_d \quad \textbf{but} \quad \mathbb{P}_x(\text{extinction in } B) = \mathbb{P}_x(\text{extinction in } A_B),$$

for all $x \in \mathbb{T}_d$. (E.g., almost sure extinction in $B \Rightarrow$ almost sure extinction in A_B even if $\partial A_B = \partial \mathbb{T}_d$.)

Thank you for your attention!