A phase transition for the tree-builder random walk

Guillaume Conchon--Kerjan

King's College London

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joint work with Arthur Blanc-Renaudie (CNRS Rouen), Camille Cazaux (LPSM Paris), Tanguy Lions (UMPA Lyon) and Arvind Singh (CNRS Paris-Saclay)

1. Model: a random walk on a growing tree

2. Results

3. Proof ideas: local times and urn processes

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3. Proof ideas: local times and urn processes

The (unbiased) model

• ν probability distribution on $\mathbb{N}$ $(:= \{0, 1, \dots\})$, $\nu(0) \neq 0$.
$\hookrightarrow (\xi_n)_{n \geq 0}$ i.i.d.$\sim \nu$

• a growing tree $T = (T_0 T_1, \dots) \to T_n$ at time n.
$\hookrightarrow T_0 = \{\circ\}$ reduced to a single vertex, the root.

• A random walk $(S_n)_{n \geq 0}$. Initially, $S(0) = \circ$

The dynamics

Given (T_n, S_n):

$T_{n+1} = T_n$ with ξ_n leaves attached to the vertex S_n.

$S_{n+1} =$ uniform random neighbour of S_n in T_{n+1} ($S_{n+1} = S_n$ if $T_{n+1} = \{\circ\}$).

Let: $|T_n|$ the height of T_n, $|S_n| = \mathrm{dist}(\circ, S_n)$ the height of S_n.
$\deg(S_n)$:=degree of S_n.

The (unbiased) model

Example: $\xi_0 = 2, \xi_1 = 3, \xi_2 = 0.$

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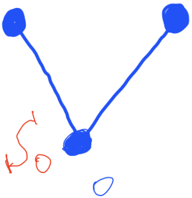

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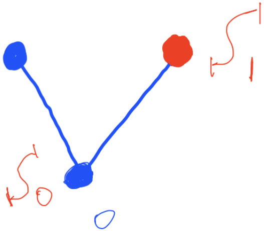

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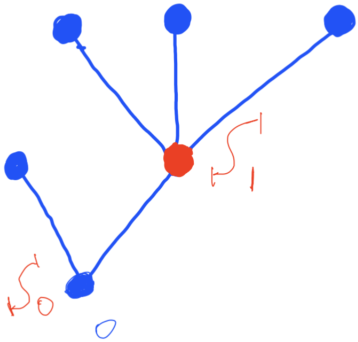

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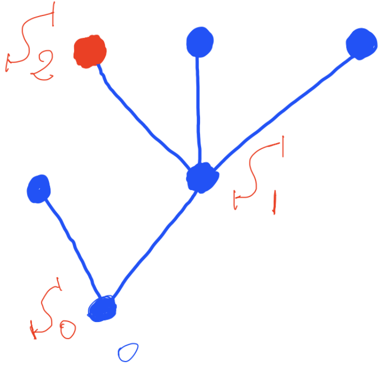

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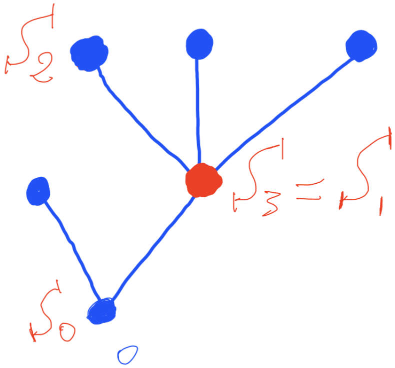


$|S_0| = 0$, $|S_1| = 1$, $|S_2| = 2$, $|S_3| = 1$.

Remarks: When (S_n) comes back to an already visited vertex, it can still add new leaves there! Also, (S_n) is not Markovian, but (T_n, S_n) is.

Questions

- Transience, *recurrence* depending on ν?

$\hookrightarrow$ Defined as returning finitely/*infinitely* many times to $\circ$.

- By the way, is there still a $0-1$ law for transience/recurrence?

Previous results

Thms (Engländer, Figuereido, Iacobelli, Oliveira, Ribeiro, Valle, Zuaznábar '21 -'24):

for every ν, there exists $v = v_\nu > 0$ s.t. $\frac{S_n}{n} \to v$ a.s.

+CLT, renewal structure.

- No phase transition - transience is too strong, even if $\nu = \mathrm{Ber}(\varepsilon)$, $\varepsilon > 0$!

$\hookrightarrow$ Can we rebalance this?

The biased model

Each vertex x has a **parent** $p(x)$, i.e. its neighbour towards the root $\circ$ ($p(\circ) = \circ$: add an oriented *root loop* to $\circ$)

The biased dynamics

T_{n+1} defined as before.

$$S_{n+1} = \begin{cases} p(S_n) & \text{with proba } \frac{\rho}{\rho+\deg(S_n)} \\ \text{any other neighbour of } S_n & \text{with proba } \frac{1}{\rho+\deg(S_n)}. \end{cases}$$

New question: if ρ is large enough, can (S_n) visit $\circ$ infinitely many times with positive proba/almost surely?

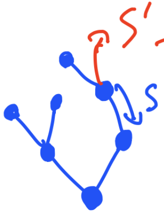
The biased model:first heuristics

New question: if ρ is large enough, can (S_n) visit $\circ$ infinitely many times with positive proba/almost surely?

Remark (coupling): transience probability decreasing with ρ and increasing with ν (w.r.t. stochastic domination).

$S \to (\rho, \nu)$, $S' \to (\rho', \nu')$ $\rho > \rho'$

$\nu' \geq_{\text{stoch.}} \nu$



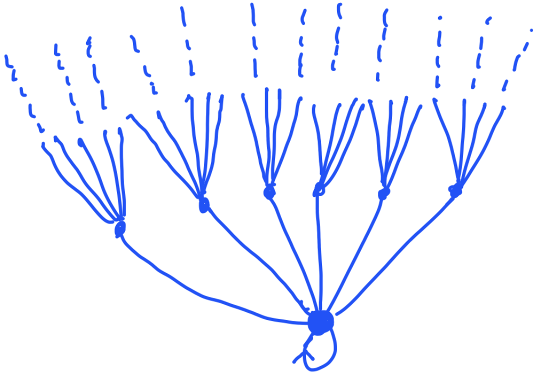
excursion of S' while S frozen

$\hookrightarrow$ We can only have one phase transition.

The biased model:first heuristics

New question: if ρ is large enough, can (S_n) visit $\circ$ infinitely many times with positive proba/almost surely?

No because...



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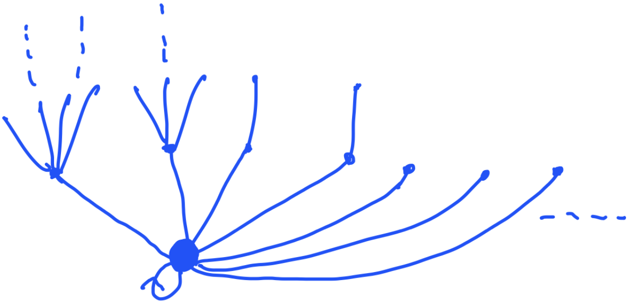
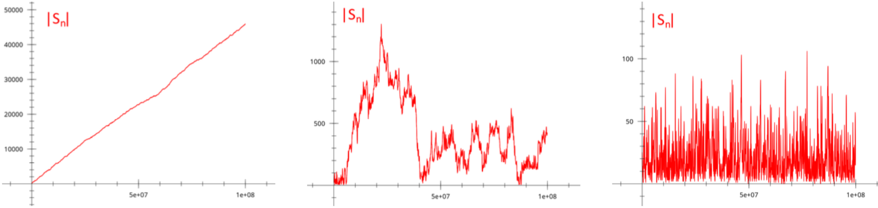
Yes because...



The biased model:first heuristics

Simulations: $\nu = \mathrm{Ber}(p)$, $p \in (0,1]$.

$p = 1$:

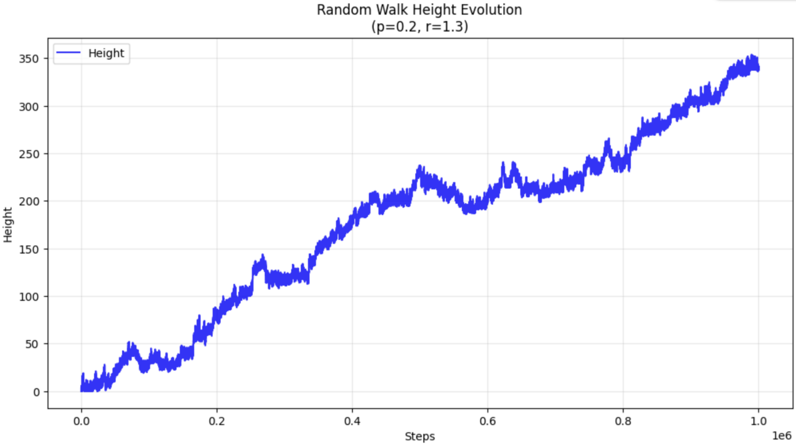


(a) $\rho = 2.95$ (b) $\rho = 3$ (c) $\rho = 3.05$

The biased model:first heuristics

Simulations: $\nu = \mathrm{Ber}(p)$, $p \in (0, 1]$.

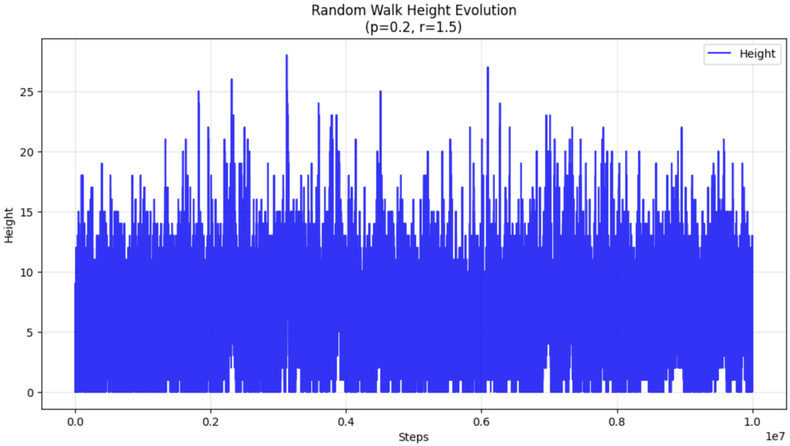
$p = .2$, $\rho = 1.5$:



The biased model:first heuristics

Simulations: $\nu = \mathrm{Ber}(p)$, $p \in (0,1]$.

$p = .2\, \rho = 1.3$:



Result - the walk (S_n)

Define $\bar{\nu} := \sum_{k\geq 1} k\nu(k) \in (0,\infty]$ the expectation of ν.

$\tau_i :=$ i-th return to the root, $i \geq 0$.

Theorem (Blanc-Renaudie, Cazaux, C., Lions, Singh '24)

If $\rho < 1 + 2\bar{\nu}$: (S_n) transient, $|S_n|/n \to v_{\rho,\nu} > 0$ a.s. (+CLT)

If $\rho \geq 1 + 2\bar{\nu}$: (S_n) is recurrent, returns infinitely many times to $\circ$ a.s.

- $\rho > 1 + 2\bar{\nu}$: 'positive recurrent' ($\mathbb{E}[\tau_i - \tau_{i-1}] < \infty$ for all $i \geq 1$)
- $\rho = 1 + 2\bar{\nu}$: 'null recurrent' ($\mathbb{E}[\tau_i - \tau_{i-1}] = \infty$ for all $i \geq 1$)

$\hookrightarrow$ Always transient when $\bar{\nu} = \infty$.

$\hookrightarrow$ Even in the positive recurrent case, $\tau_i \geq e^{\delta i}$ a.s. for i large enough, $\delta = \delta(\rho,\nu) > 0$: $\sharp\{\text{returns to } \circ\}$ grows logarithmically.

Result - the tree T_n

Theorem (Blanc-Renaudie, Cazaux, C., Lions, Singh '24)

If $\rho < 1 + 2\bar{\nu}$: $|T_n|/n \to v_{\rho,\nu}$ a.s. (+same CLT as $|S_n|$)

If $\rho > 1 + 2\bar{\nu}$: $|T_n|/\log(n) \in [c_1(\rho,\nu), c_2(\rho,\nu)]$ for n large a.s. $(c_1, c_2 > 0)$

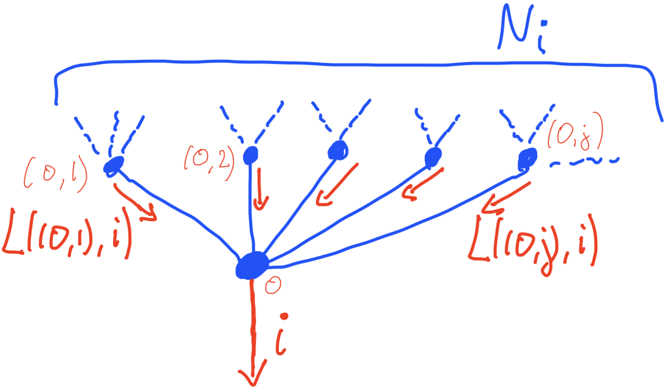
$\hookrightarrow$ conjecture when $\rho = 1 + 2\bar{\nu}$:
$|T_n| = \Theta(n^{1/2})$ (same for typical values of $|S_n|$).

Local times

$\mathcal{T}_{full}$:= usual Ulam-Harris tree rooted at $\circ$: each vertex x has countably many children, (x, j) := j-th child of x.

$L(x, i)$:= the local time at vertex $x \in \mathcal{T}_{full}$ before time $\tilde{\tau}_i$:= i-th crossing of the *root loop*.

N_i := $\sharp\{$children of $\circ$ created before $\tilde{\tau}_i\}$



Branching process of local times

Markovian structure of local times since at least Kesten - Kozlov - Spitzer '75.

Multitype branching process Z on $\mathcal{T}_{full}$, types in $\mathbb{N}$ denote the local time. Kernel $K : \mathbb{N} \mapsto \mathbb{N}^{\mathbb{N}}$.
$\hookrightarrow$ Given $Z(\circ) = i \geq 1$, offspring local times distributed as $K(i, \cdot) \overset{(d)}{=} (L((\circ, 1), i), L((\circ, 2), i), \ldots)$.

Proposition(From Basdevant - Singh '19)

(S_n) recurrent $\Leftrightarrow$ Z dies out for every initial value of $Z(\circ)$

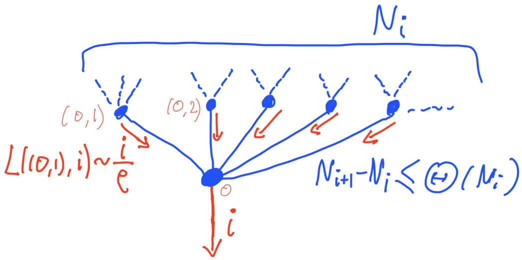
- $\mathbb{P}(\tilde{\tau}_i < \infty) = \mathbb{P}(Z \text{ dies out } \mid Z(\circ) = i)$
- Local time process $L(\cdot, i)$ under $\mathbb{P}(\cdot | \tilde{\tau}_i < \infty)$ distributed as Z under $\mathbb{P}(\cdot | Z(\circ) = i, Z \text{ dies out })$

Recurrence for large ρ

Recall $N_i := \sharp\{\text{children of } \circ \text{ created before } \tilde{\tau}_i\}$

Idea: (S_n) recurrent if there is a positive map f s.t.

$$f(i) > \mathbb{E}\left[\sum_{j=1}^{N_i} f(L((v,j),i))\right]$$



$N_i \simeq e^{c(\nu)i}$, $L((v,j),i) \lesssim \rho/i$... $f(y) = e^{Cy}$ for C large should work!

Recurrence for large ρ

Issue: recurrence for large ρ only (unicity of phase transition by monotonicity in ρ and ν), no explicit critical point/asymptotics on return times.
If $M_{i,j} :=$ expected number of offspring of type j for a particle of type i, we have f such that $f(i) > \sum_{j\geq 1} M_{i,j} f(j)$ (right "almost" eigenvector with eigenvalue < 1).

Finer approach: urns and left-eigenvalues

We had f such that $f(i) > \sum_{j\geq 1} M_{i,j} f(j)$...

... But in fact $f(i) := \rho^{-i}$ is a left eigenvector!

$$\sum_{i\geq 1} \rho^{-i} M_{i,j} = \frac{\bar{\nu}}{\rho - \bar{\nu} - 1} \rho^{-j} (\star)$$

$$(S_n) \text{ transient} \Leftrightarrow \frac{\bar{\nu}}{\rho - \bar{\nu} - 1} > 1 \Leftrightarrow \rho < 2\bar{\nu} + 1.$$

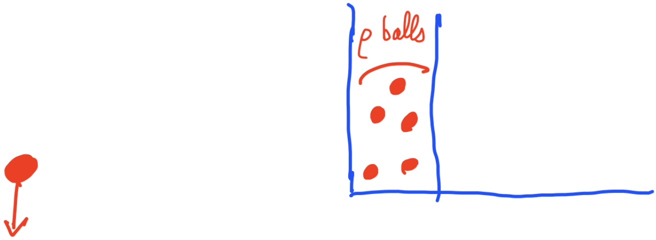
Key to prove $(\star)$: computing $\mathbb{E}[N_i - N_{i-1}]$.

(Actually $\sum_{i\geq 1} f(i) M_{i,j} = \sum_{l\leq i} f(G_l)\mathbb{E}[N_l - N_{l-1}]$ for some $\sum$s of geometric variables G_l)

Finer approach: urns and left-eigenvalues

Recall $\mathbb{E}[N_i - N_{i-1}] =$
$\sharp\{$offspring of $\circ$ created between $i-1$th and ith visit to root loop$\}$.

S moving out of $\circ \leftrightarrow$ picking from an urn of the vertices.



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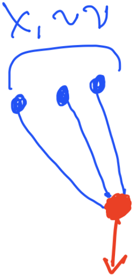

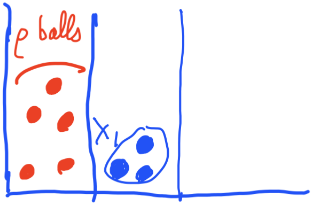

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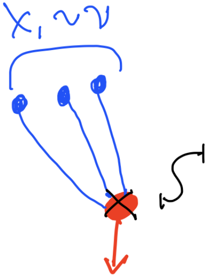

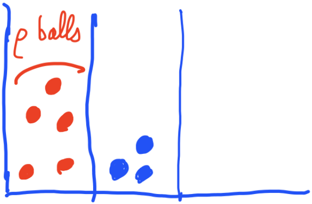

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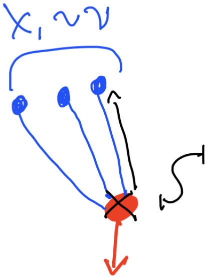

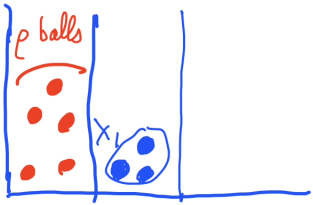

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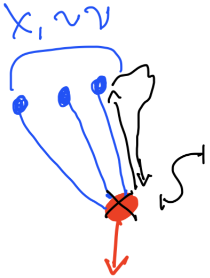

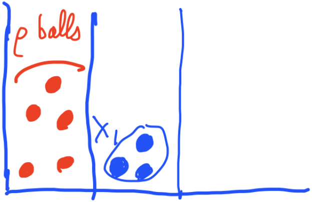

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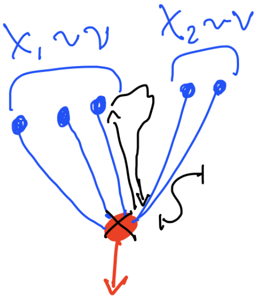

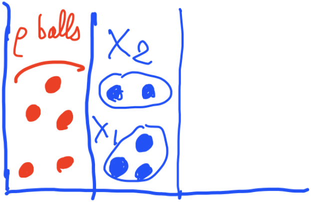

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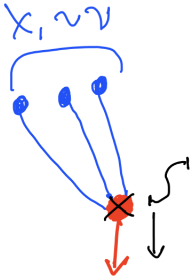

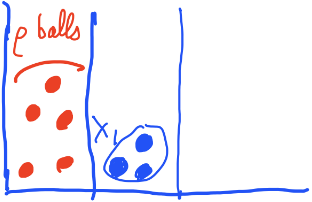

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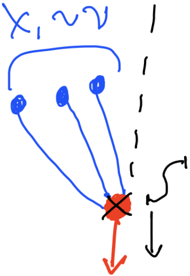

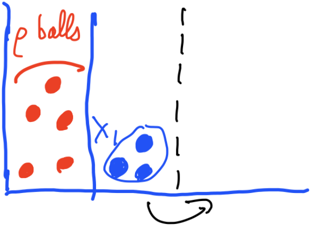

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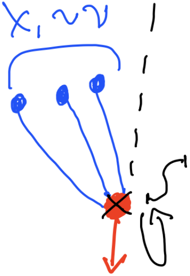

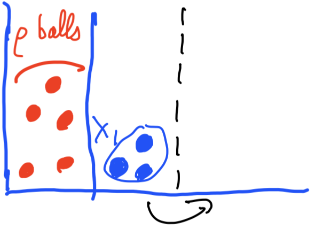

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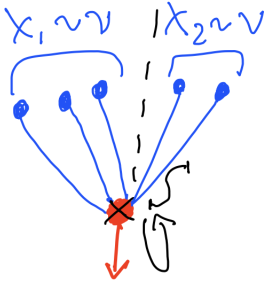

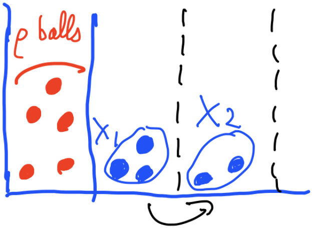

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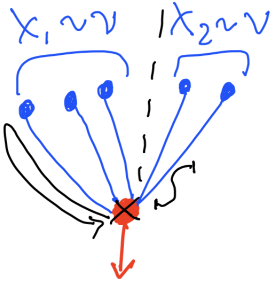

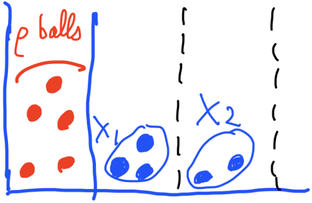

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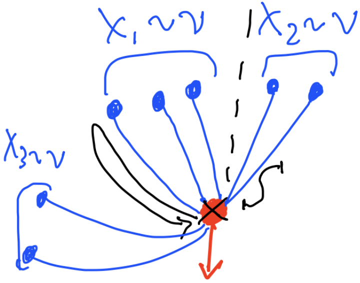

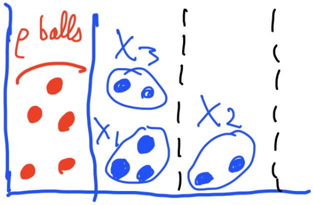

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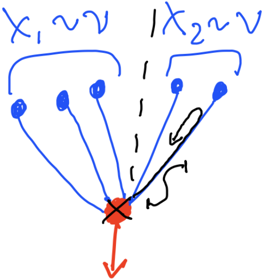

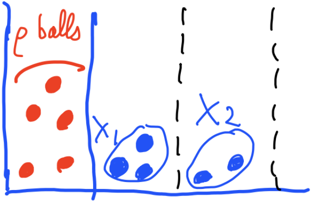

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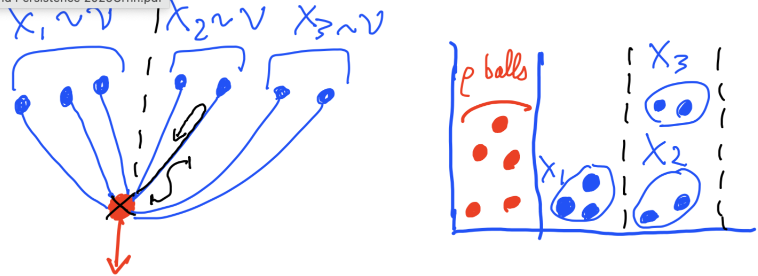
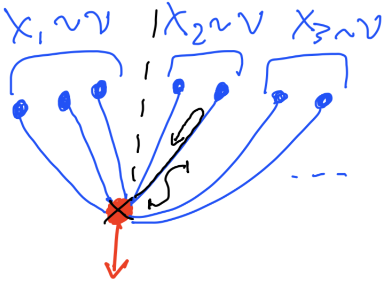
Branching and Persistence 2025Urnn.pdf

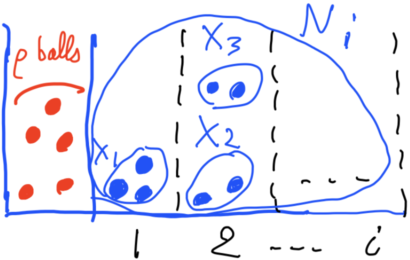


Finer approach: urns and left-eigenvalues

Recall $\mathbb{E}[N_i - N_{i-1}] =$
$\sharp\{$offspring of $\circ$ created between $i-1$th and ith visit to root loop$\}$.

S moving out of $\circ \leftrightarrow$ picking from an urn of the vertices.

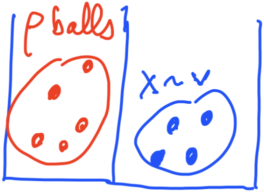




$N_i \sim \sharp\{\bullet$ added in each column until time $\tilde{\tau}_i\}$ Let's look at the j-th colum, $j \leq i$.

The Poissonized urn

Urn for the j-th column:
- start with $X \sim \nu$ $\color{blue}\bullet$, and ρ $\color{red}\bullet$.
- any ball independently picked after Exp(1) time; if $\color{blue}\bullet$: add $X' \sim \nu$ $\color{blue}\bullet$.
- stop at time t_{i-j} when we pick $i - j$ $\color{red}\bullet$.



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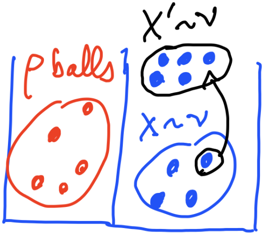

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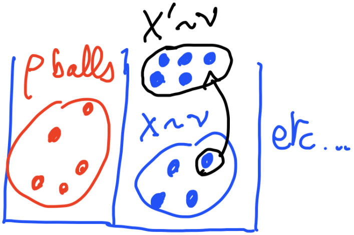

$C_t := \sharp\{\color{blue}\bullet$ added by time $t\}$.

$d\mathbb{E}[C_t]/dt = \bar{\nu}\mathbb{E}[C_t] \Rightarrow \mathbb{E}[C_t] = \bar{\nu}e^{\bar{\nu}t}$. Integrate against $t_{i-j} \sim \Gamma(i-j,\rho)$.

$\Rightarrow \mathbb{E}[C_{t_{i-j}}] = \bar{\nu}(\frac{\rho}{\rho-\bar{\nu}})^{i-j}$. Sum over j to get $\mathbb{E}[N_i]$.

Open questions

- critical case $\rho = 1 + 2\overline{\nu}$: conjecture $h(T_n), S_n = \Theta(n^{1/2})$

- monotonicity of speed in ν and ρ?

- vary ν in time, e.g. $\xi_n \sim \mathrm{Ber}(n^{-\gamma})$. (S_n) recurrent if $\gamma > 1/2$ (Engländer, Iacobelli, Ribeiro '23), conjectured transient if $\gamma < 1/2$.

- graph builder random walk: S_n can be connected to pre-existing vertices instead of new leaves

Thank you for your attention!