Number of particles at sublinear distances from the tip in branching Brownian motion

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Branching and Persistence, Angers, April 2025

Oxford Mathematics

Branching Brownian motion

Branching Brownian motion (BBM) combines Brownian motion and branching processes. At time $t = 0$, a particle starting in $0 \in \mathbb{R}$ moves as Brownian motion. After a random time exp(1) it splits in 2. Each particle repeats this process independently from the splitting position.

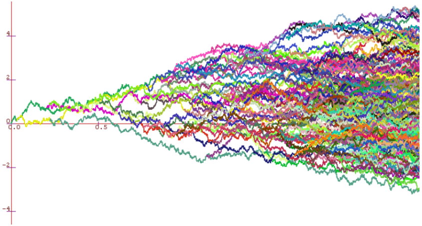

Figure: Path of the BBM particles, Matthew Roberts

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Lalley and Sellke [1] proved that there exists a constant $C > 0$ such that

$$M_t - m_t - \log(CZ_\infty)/\sqrt{2} \xrightarrow[t\to\infty]{d} G, \tag{1}$$

where $m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$, G follows a Gumbel distribution, and $Z_\infty > 0$ is the almost sure limit of the derivative martingale:

$$Z(t) = \sum_{u \in \mathcal{N}_t} (\sqrt{2}t - X_u(t)) e^{\sqrt{2}(X_u(t) - \sqrt{2}t)}.$$

Theorem 1

For $t \geq 0$, denote the number of particles to the right of $m_t - x$,

$$N(t,x) = \#\{u \in \mathcal{N}_t : X_u(t) \geq m_t - x\}, \tag{2}$$

Theorem

Let x_t be such that, as $t \to \infty$, $x_t = o_t(t/\log(t))$ and $x_t \to \infty$, then,

$$\frac{N(t,x_t)}{\pi^{-\frac{1}{2}} x_t e^{\sqrt{2}x_t} e^{-\frac{x_t^2}{2t}}} \xrightarrow[t\to\infty]{\mathbb{P}} Z_\infty. \tag{3}$$

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Denote $f(t,x) := \pi^{-\frac{1}{2}} x e^{\sqrt{2}x} e^{-\frac{x^2}{2t}}$.

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The statement of Theorem 1 is equivalent to, for any $\delta > 0$,

$$\lim_{x\to\infty} \lim_{c\to 0} \sup_{t: x \leq c\frac{t}{\ln(t)}} \mathcal{P}\left(\left|\frac{N(t,x)}{f(t,x)} - Z_\infty\right| > \delta\right) = 0. \tag{4}$$

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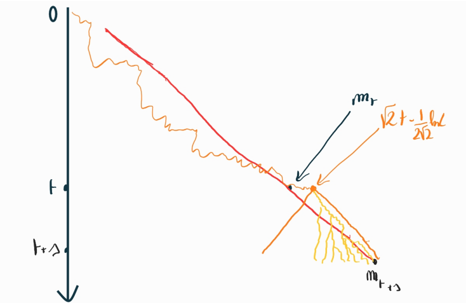

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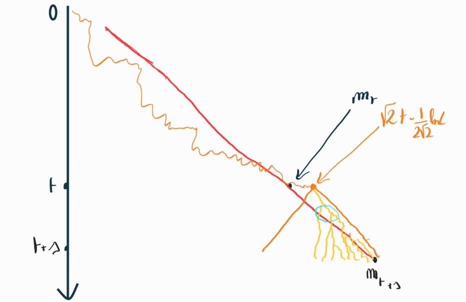

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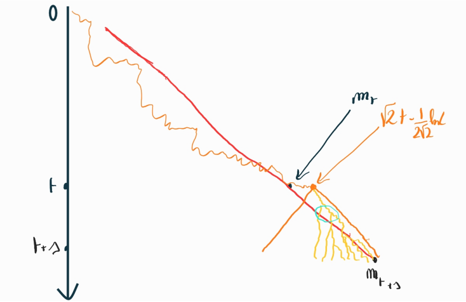

$$\sqrt{2}t - \frac{1}{2\sqrt{2}} \log t + m_{s_\star} = m_{t+s_\star}$$
$$s_\star \approx t^{2/3}$$

Theorem 3

Define $Q_t(u,v) = \sup\{s \leq t \text{ such that } \forall \gamma \leq s : X_u(s) = X_v(s)\}$, for $(u,v) \in \mathcal{N}_t^2$.

Theorem

Let x_t and R_t be such that, as $t \to \infty$, $x_t = o_t(t/\log(t))$, $x_t \to \infty$ and $R_t \to \infty$. Then for any $\delta > 0$:

$$\lim_{t \to \infty} \mathbb{P}\left(\frac{\#\{(u,v) \in \mathcal{N}(t,x_t)^2 : Q_t(u,v) > R_t\}}{f(t,x_t)^2} > \delta \right) = 0. \tag{6}$$

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Equivalently, by Theorem 1, let U and V be chosen uniformly in $\mathcal{N}(t,x_t)$ and define $Q_t = Q_t(U,V)$ if $\mathcal{N}(t,x) \neq \emptyset$ and $Q_t = t$ otherwise. Then we have that $(Q_t)_{t\in\mathbb{R}}$ is tight, i.e.

$$\lim_{R\to\infty} \sup_{t\geq 0} \mathbb{P}(Q_t > R) = 0. \tag{7}$$

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- Estimate the conditional expectation of the number of localized particles given the initial behavior of the BBM.
- Prove that this conditional expectation is close to the number of localized particles via a concentration argument.

Conditional expectation estimation

Lemma

As $t \to \infty$, for x_t such that $x_t = o(t \log(t)^{-1})$ and $x_t \to \infty$, for r such that $r \to \infty$ and for R such that $R = o(\sqrt{t})$, $R = o(t/x_t)$, and $R > r$, the following holds:

$$\mathbb{E}\left[\#\{u \in \mathcal{N}_t : X_u(t) \geq m_t - x_t \text{ and } X_u(\cdot) \text{ is loc after } r\} \mid \mathcal{F}_R\right] = f(t, x_t) Z_\infty (1 + o_t(1)) \quad a.s, \tag{8}$$

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Denote $\mathcal{N}(t, X, r) = \{u \in \mathcal{N}_t : X_u(t) \geq m_t - X \text{ and } X_u(\cdot) \text{ is loc after } r\}$, $N(t, X, r) = \#\mathcal{N}(t, X, r)$.

Link between Theorem 1 and Theorem 3

Lemma

For $t, 0 \leq R \leq t, X \in \mathbb{R}$*,*

$$\mathbb{E}\left[\left(N(t,X,r) - \mathbb{E}\left[N(t,X,r) \mid \mathcal{F}_R\right]\right)^2\right] \leq \mathbb{E}\left[\#\{(u,v) \in \mathcal{N}(t,X,r)^2 : Q_t(u,v) > R\}\right] \tag{9}$$

Open problems

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Let x_t be such that, as $t \to \infty$, $x_t = o_t(t/\log(t))$ and $x_t >> \sqrt{t}$, then,

$$\frac{N(t, x_t)}{f(t, x_t)} \xrightarrow[t\to\infty]{} Z_\infty \quad a.s. \tag{10}$$

S. P. Lalley and T. Sellke.
A conditional limit theorem for the frontier of a branching Brownian motion.
Ann. Probab., 15:1052–1061, 1987.