Asymptotics of the Overlap Distribution of Branching Brownian Motion

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Contents

Branching Brownian motion (BBM)

Definition:

- At time $t = 0$, a particle starts a Brownian motion in $\mathbb{R}$.
- After a random time with exponential distribution, it dies and gives birth to two children.
- Each child then repeats this process independently of each other.

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Motivations:

- Discrete counterpart: branching random walk.
- Toy model for spin glass systems or log-correlated field.
- Deep connection with the F-KPP equation.

Branching Brownian motion (BBM)

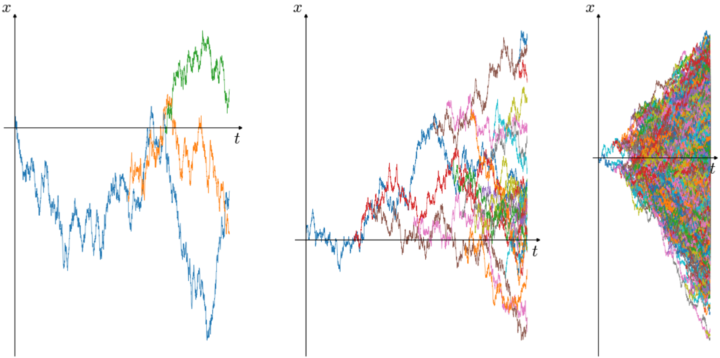

Figure: Realizations of a BBM over three different times.

Additive and derivative martingales

Gibbs measure:

- $\mathcal{N}(t)$: set of particles alive at time t.
- $X_u(t)$: position of particle u at time t.
- $\mathcal{G}_{\beta,t}(u) \propto \mathrm{e}^{\beta X_u(t)}$.

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Martingales:

- *Additive martingale* at inverse temperature $\beta \geq 0$,

$$W_t(\beta) = \mathrm{e}^{-\psi(\beta)t} \sum_{u \in \mathcal{N}(t)} \mathrm{e}^{\beta X_u(t)}, \quad \text{where } \psi(\beta) = 1 + \beta^2/2.$$

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- *Derivative martingale* at inverse temperature $\beta \geq 0$,

$$Z_t(\beta) = \mathrm{e}^{-\psi(\beta)t} \sum_{u \in \mathcal{N}(t)} (\beta t - X_u(t))\mathrm{e}^{\beta X_u(t)}.$$

Phase transition

Since $W_t(\beta)$ is a non-negative martingale, it converges a.s. to some $W_\infty(\beta)$.

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Theorem ([Kingman, 1975, Biggins, 1977, Neveu, 1988])

A phase transition occurs at $\beta_c = \sqrt{2}$. More precisely,

- *if $\beta \geq \beta_c$, then $W_\infty(\beta) = 0$ a.s.,*
- *if $\beta < \beta_c$, then $W_t(\beta)$ is uniformly integrable and $W_\infty(\beta) > 0$ a.s.*

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Theorem ([Lalley and Sellke, 1987])

The critical derivative martingale $Z_t := Z_t(\beta_c)$ converges a.s. to some $Z_\infty > 0$.

The overlap

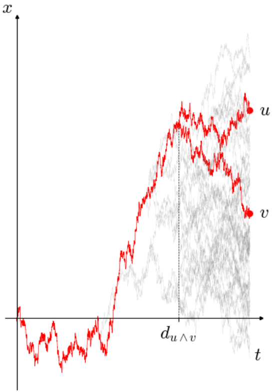

Figure: Time of death of the last common ancestor of two particles.

The overlap

Overlap $q_t(u, v)$ of two particles $u, v \in \mathcal{N}(t)$: age of their last common ancestor, rescaled by a factor of t.

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Overlap $q_t(u,v)$ of two particles $u, v \in \mathcal{N}(t)$: age of their last common ancestor, rescaled by a factor of t.

Question: Where is the overlap of two particles chosen independently at random according to $\mathcal{G}_{\beta,t}$?

We want to understand the asymptotic behavior of the *overlap distribution*, defined for $a \in (0,1)$ by

$$\nu_{\beta,t}([a,1]) = \left\langle \mathbb{1}_{\{q_t(u,v)\geq a\}} \right\rangle_{\beta,t} = \frac{\sum_{u,v\in\mathcal{N}(t)} \mathrm{e}^{\beta(X_u(t)+X_v(t))}\mathbb{1}_{\{q_t(u,v)\geq a\}}}{\sum_{u,v\in\mathcal{N}(t)} \mathrm{e}^{\beta(X_u(t)+X_v(t))}}.$$

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Typical overlap distribution

Theorem 1 ([C. and Pain, 2024])

Let $a \in (0,1)$. *As* $t \to \infty$,

1. *if* $0 \leq \beta < \beta_c/2$, *then*

$$\mathrm{e}^{(1-\beta^2)at}\nu_{\beta,t}([a,1]) \xrightarrow{a.s.} \frac{W_\infty(2\beta)}{W_\infty(\beta)^2}\mathbb{E}\left[W_\infty(\beta)^2\right],$$

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2. *if* $\beta = \beta_c/2$, *then*

$$\sqrt{at}e^{at/2}\nu_{\beta,t}([a,1]) \xrightarrow{\mathbb{P}} \sqrt{\frac{2}{\pi}}\frac{Z_\infty}{W_\infty(\beta)^2}\mathbb{E}\big[W_\infty(\beta)^2\big],$$

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3. *if* $\beta_c/2 < \beta < \beta_c$*, then*

$$(at)^{3\beta/\beta_c}\mathrm{e}^{(\beta_c-\beta)^2 at}\nu_{\beta,t}([a,1]) \xrightarrow{(\mathrm{d})} \frac{(Z_\infty)^{2\beta/\beta_c}}{W_\infty(\beta)^2}S,$$

where S *is* $\beta_c/2\beta$*-stable and independent of the BBM.*

Sketch of proof for $0 \leq \beta < \beta_c/2$

- Rewrite the overlap distribution

$$\begin{aligned}
\nu_{\beta,t}([a,1]) &= \frac{e^{-(1-\beta^2)at}}{W_t(\beta)^2} \sum_{w \in \mathcal{N}(at)} e^{2\beta X_w(at) - \psi(2\beta)at} W_{t-at}^{(w,at)}(\beta)^2 \\
&\approx \frac{e^{-(1-\beta^2)at}}{W_t(\beta)^2} \underbrace{\sum_{w \in \mathcal{N}(at)} e^{2\beta X_w(at) - \psi(2\beta)at}}_{=W_{at}(2\beta)} \mathbb{E}\left[W_\infty(\beta)^2\right] \\
&\sim e^{-(1-\beta^2)at} \frac{W_\infty(2\beta)}{W_\infty(\beta)^2} \mathbb{E}\left[W_\infty(\beta)^2\right], \qquad \text{a.s.}
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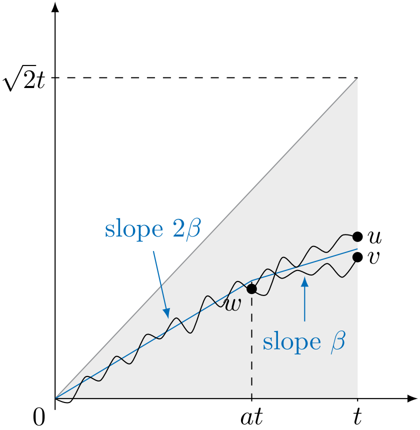
$$\begin{aligned}\nu_{\beta,t}([a,1]) &= \frac{\mathrm{e}^{-(1-\beta^2)at}}{W_t(\beta)^2} \sum_{w \in \mathcal{N}(at)} \mathrm{e}^{2\beta X_w(at) - \psi(2\beta)at} W_{t-at}^{(w,at)}(\beta)^2 \\ &\approx \frac{\mathrm{e}^{-(1-\beta^2)at}}{W_t(\beta)^2} \underbrace{\sum_{w \in \mathcal{N}(at)} \mathrm{e}^{2\beta X_w(at) - \psi(2\beta)at}}_{=W_{at}(2\beta)} \mathbb{E}\left[W_\infty(\beta)^2\right] \\ &\sim \mathrm{e}^{-(1-\beta^2)at} \frac{W_\infty(2\beta)}{W_\infty(\beta)^2} \mathbb{E}\left[W_\infty(\beta)^2\right], \qquad \text{a.s.}\end{aligned}$$

- Particles that contribute mainly to $\nu_{\beta,t}([a,1])$ follow a Brownian motion with drift 2β until time at and then with drift β.

Typical trajectories



(a) Case $0 \leq \beta < \beta_c/2$. Particles that contribute to $\nu_{\beta,t}([a,1])$ have drift 2β until time at and then have drift β.

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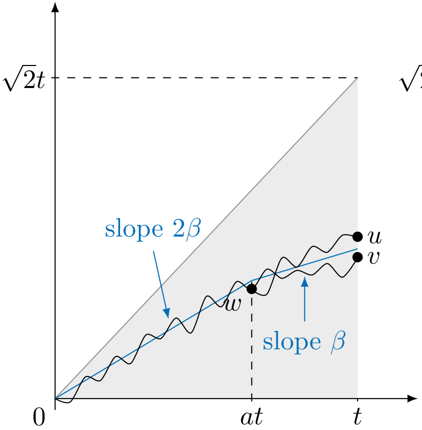

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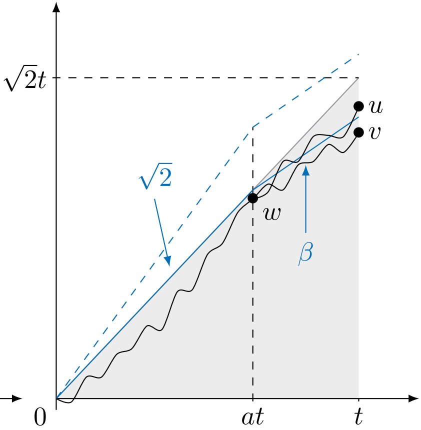

(b) Case $\beta_c/2 < \beta < \beta_c$. Particles that contribute to $\nu_{\beta,t}([a,1])$ are near the top at time at and then have drift β.

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Expected overlap distribution

Theorem 2 ([C. and Pain, 2024])

Let $a \in (0,1)$. *As* $t \to \infty$,

1. *if* $\beta = 0$, *then*

$$\mathbb{E}[\nu_{\beta,t}([a,1])] \sim 2ate^{-at},$$

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1. *if* $\beta = 0$*, then*
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$$\text{“}\ \mathbb{E}[\nu_{\beta,t}([a,1])] \sim \mathrm{e}^{-(1-\beta^2)at}\mathbb{E}\left[\frac{W_\infty(2\beta)}{W_\infty(\beta)^2}\right]\mathbb{E}\left[W_\infty(\beta)^2\right]\ \text{”},$$

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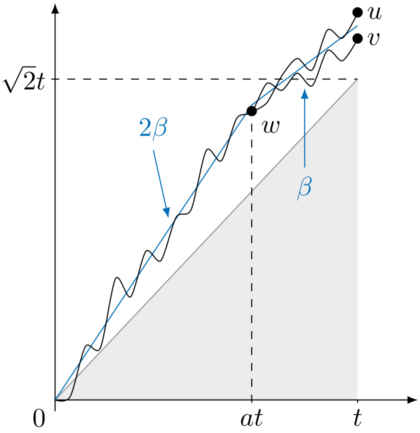
Trajectories contributing to the expectation



(a) Case $0 \leq \beta < \sqrt{2/3}$. Particles that contribute to $\mathbb{E}\left[\nu_{\beta,t}([a,1])\right]$ have drift 2β until time at and then have drift β.

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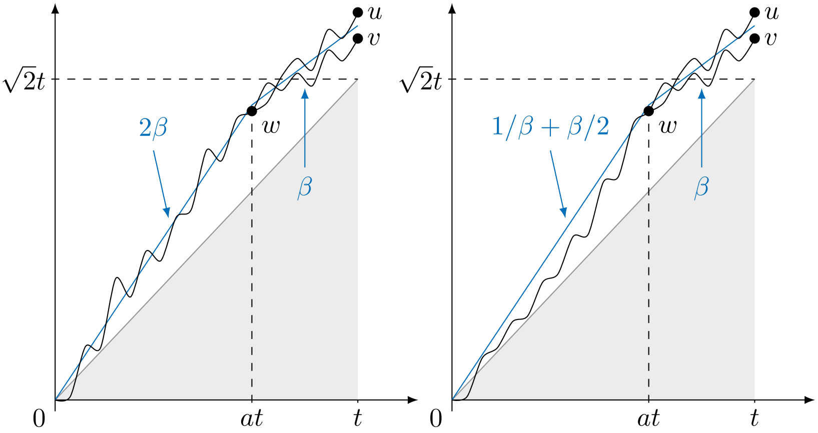

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(b) Case $\sqrt{2/3} < \beta < \beta_c$. Particles that contribute to $\mathbb{E}\big[\nu_{\beta,t}([a,1])\big]$ are near $(1/\beta + \beta/2)at$ at time at and then have drift β.

Perspectives

- What about $\beta = \beta_c$ and $\beta > \beta_c$?
- Overlap at two temperatures $\beta \neq \beta'$.
- Asymptotics of $\nu_{\beta_t,t}$ when $\beta_t \to \beta_c/2$, of $\mathbb{E}\nu_{\beta,t}$ when $\beta_t \to \sqrt{3/2}$.
- Generalization: random offspring distribution, other branching models (e.g. CREM).

Thank you for listening!

Appendix: Trajectories contributing to the expectation

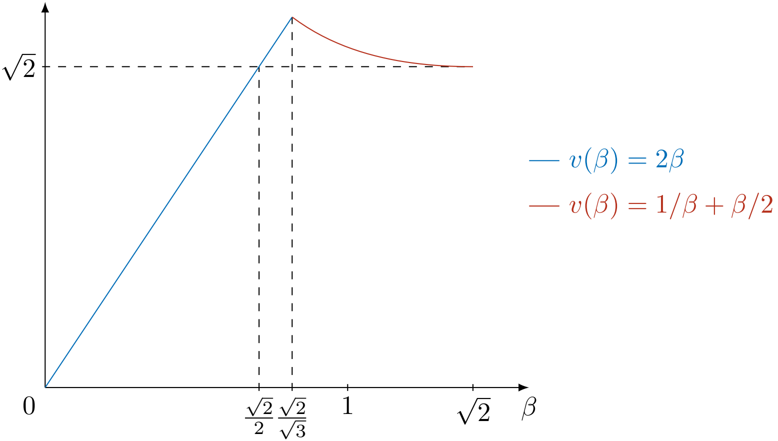

Figure: Graph of the function v such that particles that contribute to $\mathbb{E}[\nu_{\beta,t}([a,1])]$ are "near" $v(\beta)at$ at time at.

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