Persistence probabilities of spherical fractional Brownian motion

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Angers, 10th April 2025

Outline

Definition of Fractional Brownian Motion

For $0 < H < 1$:
Continuous, centered Gaussian Process $(B_H(t))_{t\in\mathbb{R}}$ on $\mathbb{R}$ with covariance

$$\frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right)$$

for $s, t \in \mathbb{R}$.

Properties

- Increments independent if, and only if, $H = 1/2$
- H-self similar, i.e. $B_H(at) \stackrel{fdd}{=} |a|^H B_H(t)$
- Stationary increments, i.e. $B_H(t) - B_H(s) \stackrel{fdd}{=} B_H(t-s)$

Persistence

- Find the asymptotic behaviour of

$$\mathbb{P}\left(\sup_{t\in[0,T]} B_H(t) < 1\right) \qquad \text{for } T \to \infty$$

- For self-similar processes we expect $T^{-\theta+o(1)}$ for $\theta \in (0,\infty)$.
- "large domain - fixed barrier"
  $\Leftrightarrow$ "fixed domain - small barrier"

$$\mathbb{P}\left(\sup_{t\in[0,1]} B_H(t) < \varepsilon\right) = \varepsilon^{\frac{\theta}{H}+o(1)} \qquad \text{for } \varepsilon \to 0$$

FBM on $\mathbb{R}^+$

Theorem (Molchan (1999))

The persistence exponent of Fractional Brownian Motion on the positive real line $(B_H(t))_{t\geq 0}$ is $1-H$, i.e.

$$\mathbb{P}\left(\sup_{t\in[0,T]} B_H(t) < 1\right) = T^{-(1-H)+o(1)} \quad \text{as } T \to \infty$$

- What happens, if we replace $[0, T]$ by $[-T, T]$?

Two-Sided FBM

Theorem (Molchan (1999))

The persistence exponent of FBM on the whole real line $(B_H(t))_{t\in\mathbb{R}}$ is 1, i.e.

$$\mathbb{P}\left(\sup_{t\in[-T,T]} B_H(t) < 1\right) = T^{-1+o(1)} \quad \text{as } T \to \infty$$

- What happens in domains that do not contain 0?

Domains without Zero

- Let $K \subset \mathbb{R}$ compact and $0 \notin K$:

$$\begin{aligned} \mathbb{P}\left(\sup_{t\in TK} B_H(t) \leq 1\right) \quad &= \dots \\ &\geq \mathbb{P}\left(\sup_{t\in K} B_H(t) \leq 0\right) \\ &> 0 \end{aligned}$$

- What happens in higher dimensions?

FBM with Multi-Dimensional Time

- Replace absolute values by Euclidean norm
- For $0 < H < 1$:
  Continuous, centered Gaussian Process $(B_H(t))_{t\in\mathbb{R}^d}$ on $\mathbb{R}^d$ with covariance

$$\frac{1}{2}\left(\|s\|^{2H} + \|t\|^{2H} - \|t-s\|^{2H}\right)$$

for $s, t \in \mathbb{R}^d$

FBM with Multi-Dimensional Time

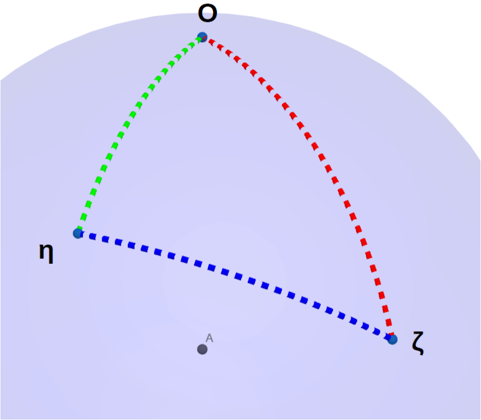
Theorem (Molchan (1999))

The persistence exponent of FBM $(B_H(t))_{t\in\mathbb{R}^d}$ on the real d-dimensional space is d, i.e.

$$\mathbb{P}\left(\sup_{s\in[-T,T]^d} B_H(s) < 1\right) = T^{-d+o(1)} \quad \text{as } T \to \infty$$

- What happens with FBM on different geometry?

FBM on the Sphere (SFBM)



FBM on the Sphere (SFBM)

- Replace Euclidean norm by geodesic / angular distance $d(.,.)$
- Specify an origin $O = (1, 0, \dots, 0)$
- For $0 < H \leq 1/2$ (Istas (2005)):
  Continuous, centered Gaussian Process $(S_H(\eta))_{\eta \in \mathbb{S}_{d-1}}$ on $\mathbb{S}_{d-1}$ with covariance

$$\frac{1}{2}\left(d(\eta, O)^{2H} + d(\zeta, O)^{2H} - d(\eta, \zeta)^{2H}\right)$$

  for $\eta, \zeta \in \mathbb{S}_{d-1}$.
- Not a covariance function for $H > 1/2$ (Istas (2005))

Persistence Exponent of SFBM

Theorem (Aurzada, H. (2025))

The persistence exponent of SFBM $(S_H(\eta))_{\eta \in \mathbb{S}_{d-1}}$ on the sphere $\mathbb{S}_{d-1}$ is equal to $d-1$, i.e.

$$\mathbb{P}\left(\sup_{\eta \in \mathbb{S}_{d-1}} S_H(\eta) < \varepsilon\right) = \varepsilon^{\frac{(d-1)}{H} + o(1)} \quad \textit{as } \varepsilon \to 0.$$

Ingredients for the Proof of the Lower Bound

- A geometric comparison inequality (Toponogov's Theorem)
- A stochastic comparison inequality (Slepian's Lemma)
- Results for Euclidean FBM by Molchan

Toponogov's Theorem

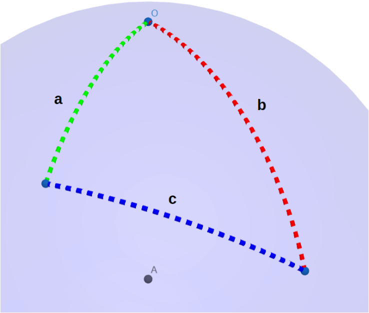


Figure: Spherical Triangle

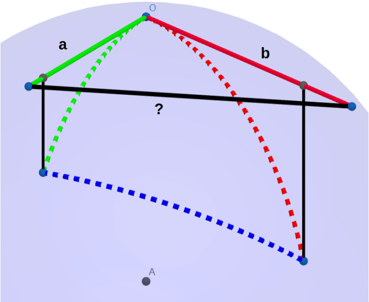


Figure: Comparison Triangle

- **Toponogov (1959):** $c \leq$ "?"
- (Cov. on $\mathbb{S}_{d-1}$) $\geq$ (Cov. on $\mathbb{R}^{d-1}$)

Slepian's Lemma

- Comparison result for Gaussian processes
- E.g. (X_t) and (Y_t) centred unit GP and $\mathbb{E}[X_s X_t] \geq \mathbb{E}[Y_s Y_t]$:

$$\mathbb{P}\left(\sup_{t\in I} X_t \leq \varepsilon\right) \overset{\text{Sl.}}{\geq} \mathbb{P}\left(\sup_{t\in I} Y_t \leq \varepsilon\right)$$

- Ordered covariance $\Rightarrow$ Ordered persistence probability

Slepian's Lemma for pos. correlated processes

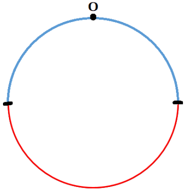
- For a pos. correlated GP $(X_t)_{t\in I}$ with $A, B \subseteq I$, we have

$$\mathbb{P}\left(\sup_{t\in A\cup B} X_t \leq \varepsilon\right) \overset{\text{Sl.}}{\geq} \mathbb{P}\left(\sup_{t\in A} X_t \leq \varepsilon\right)\mathbb{P}\left(\sup_{t\in B} X_t \leq \varepsilon\right)$$

- SFBM is positively correlated

Proof of the Lower Bound - Sketch

- Split into upper hemisphere $\mathcal{H}(O)$ and lower hemisphere $\mathcal{H}(\overline{O})$ and use Slepian's Lemma



$$\mathbb{P}\left(\sup_{\xi\in\mathbb{S}_{d-1}} S_H(\xi) \leq \varepsilon\right)$$

$$\overset{\text{Sl.}}{\geq} \mathbb{P}\left(\sup_{\xi\in\mathcal{H}(O)} S_H(\xi) \leq \varepsilon\right) \mathbb{P}\left(\sup_{\xi\in\mathcal{H}(\overline{O})} S_H(\xi) \leq \varepsilon\right)$$

- Toponogov's Theorem + Slepian's Lemma for bound on upper hemisphere

Outline

Euclidean Case

Spherical Case

Main Idea from the Proof

Future work

- Hyperbolic FBM & More

FBM on Hyperbolic Space

- Real Hyperbolic Space $\mathbb{H}_{d-1}$ has constant curvature -1
- Existence of Hyperbolic FBM (for $0 < H \leq 1/2$) by Istas (2005)
- Upper bound for Persistence (instead of lower bound) given by Toponogov's Theorem + Slepian's Lemma

FBM on Riemannian manifolds $\mathcal{M}$

Idea for persistence exponent (PE) of SFBM on $\mathcal{M}$:

$$\text{PE on } \mathbb{S}_{d-1} = d-1$$

$$\leq$$

$$\text{PE on } \mathcal{M}$$

$$\leq$$

$$\text{PE on } \mathbb{H}_{d-1} = d-1$$

Danke für die Aufmerksamkeit! :)

References

[1] F. Aurzada and M. Helmer. 'Persistence probabilities of spherical fractional Brownian motion'. 2025. arXiv: 2503.03425 [math.PR].

[2] J. Istas. 'Spherical and Hyperbolic Fractional Brownian Motion'. In: Electronic Communications in Probability 10 (2005), pp. 254–262.

[3] J. Istas. 'Karhunen–Loève expansion of spherical fractional Brownian motions'. In: Statistics & Probability Letters 76.14 (2006), pp. 1578–1583.

[4] J. Istas. 'Manifold indexed fractional fields'. In: ESAIM: PS 16 (2012), pp. 222–276.

[5] G. Molchan. 'Maximum of a fractional Brownian motion: probabilities of small values'. In: Comm. Math. Phys. 205.1 (1999), pp. 97–111.

[6] G. Molchan. 'Survival exponents for some Gaussian processes'. In: Int. J. Stoch. Anal. (2012), Art. ID 137271, 20.

[7] G. Molchan. 'Persistence exponents for Gaussian random fields of fractional Brownian motion type'. In: J. Stat. Phys. 173.6 (2018), pp. 1587–1597.

Title: *Persistence probabilities of spherical fractional Brownian motion*

URL: `https://arxiv.org/abs/2503.03425`

Brownian Bridge vs. Circular Brownian Motion

- Brownian Bridge $(BB(t))_{t\in[0,2\pi]}$:

$$BB(t) = B(t) - \frac{t}{2\pi} B(2\pi)$$

- Circular Brownian Motion (on $\mathbb{S}_1$):

$$S_{\frac{1}{2}}(t) = \begin{cases} B(t) & \text{for } t \in [0,\pi] \\ B(\pi) - B(t-\pi) & \text{for } t \in [\pi, 2\pi] \end{cases}$$

FBM on $[-T^\alpha, T]$

Theorem (Molchan (2012))

The persistence exponent of FBM $(B_H(t))_{t\in\mathbb{R}}$ on the interval $[-T^\alpha, T]$ for $0 \leq \alpha \leq 1$ is $(1-\alpha)(1-H) + \alpha \cdot 1$, i.e.

$$\mathbb{P}\left(\sup_{s\in[-T^\alpha, T]} B_H(s) < 1\right) = T^{-[(1-\alpha)(1-H) + \alpha\cdot 1]+o(1)} \quad \text{as } T \to \infty$$

FBM with Multi-Dimensional Time on Restricted Domains

Theorem (Molchan (2018))

The persistence exponent of FBM $(B_H(t))_{t\in\mathbb{R}^d}$ on domains $K := [0,1]\times[-1,1]^{d-1}$ is $d-H$, i.e.

$$\mathbb{P}\left(\sup_{s\in TK} B_H(s) < 1\right) = T^{-(d-H)+o(1)} \quad \text{as } T\to\infty$$

FBM with Multi-Dimensional Time on Restricted Domains

Conjecture

The persistence exponent of Fractional Brownian Motion $(B_H(t))_{t\in\mathbb{R}^d}$ on domains $K := [0,1]^k \times [-1,1]^{d-k}$ is $d - kH$, i.e.

$$\mathbb{P}\left(\sup_{s\in TK} B_H(s) < 1\right) = T^{-(d-kH)+o(1)} \quad \text{as } T \to \infty$$