Coexisting of branching populations

Branching and persistence 2025

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Outline

Branching process in random environment

Random environment

We consider a random environment sequence $Q = (Q(1), Q(2), \dots)$ of random probability measures on $\mathbb{N}_0$. We also consider it to be i.i.d. thus $Q(1), Q(2), \dots$ are independent copies of some random probability measure μ with generating function F on $\mathbb{N}_0$.

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Branching process

Branching processes in random environment could be thought of as developing in a scheme of two stages. First, the random environment $Q = (Q(1), Q(2), \dots)$ is established. Given its value $q = (q(1), q(2), \dots)$, the branching process $\mathcal{Z}$ then evolves in the varying environment Q. $Z(n)$ may again be realized as

$$Z(n) = \sum_{i=1}^{Z(n-1)} Y(i, n),$$

where given the environment $q(n)$, $Y(i, n)$ are random independent variables with distributions $q(n)$.

Random walks

Associated random walk

Let Q be an i.i.d. environment. We require $F[0] < 1$ and $0 < \mathbf{E}\mu < \infty$ a.s. Then we can define $X(n) := \log \mathbf{E}q(n)$ and

$$S(0) = 0, \quad S(n) = \sum_{i=1}^{n} X(n)$$

$\{S(n)\}_{n=0}^{\infty}$ is called an associated random walk.

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Critical Branching process

We say that Branching process in random environment is critical if

$$\limsup_{n\to\infty} S(n) = \infty, \ \liminf_{n\to\infty} S(n) = -\infty$$

Survival probability

Survival probability in critical case

There exists a positive finite constant $\mathcal{Y}$ such that, as $n \to \infty$:

$$\mathbb{P}\big(Z(n) > 0\big) \sim \mathcal{Y} \cdot \mathbb{P}\big(\min\{S(0), \dots S(n)\} \geqslant 0\big)$$

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Stopping time

We define

$$\tau_x := \inf\{n \geqslant 1 : \ x + S(n) \leqslant 0\} \ x \geqslant 0.$$

Then we can reformulate the result above as:

$$\mathbb{P}\big(Z(n) > 0\big) \sim \mathcal{Y} \cdot \mathbb{P}\big(\tau_0 > n\big).$$

Here we can consider stopping time τ_0 as a first time random walk $\{S(n)\}_{n=0}^{\infty}$ leaves the cone $\mathcal{K}^1 := \{y \in \mathbb{R} : \ y \geqslant 0\}$ if it starts at $S(0) = 0$.

Harmonic functions

Renewal function

In the critical case we have important result. Let $V(x)$ be a renewal function for random walk $\{S(n)\}$. Then one has $\mathbb{P}(\tau_x > n) \sim \frac{V(x)}{\sqrt{n}}$ (Doney '95).
For the renewal function in one-dimensional case main asymptotics is known

$$V(x) \sim C \cdot x, \quad x \to \infty,$$

where $C > 0$ is some constant that does not depend on x and n.

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Harmonic property

$V(x)$ is strictly positive in $\mathcal{K}^1$ and moreover for arbitrary $x \in \mathcal{K}^1$ it is true that:

$$\mathbb{E}\big[V(x+X),\ x+X \in \mathcal{K}^1\big] - V(x) = 0,$$

where X is one step of random walk. In other words, V is a strictly positive eigenfunction for operator

$$f \ \to \ \mathbb{E}f - f,$$

with eigenvalue 0.

Multi-type Branching processes in random environment

One-dimensional Branching processes in joint environment

Here we consider several population with the following properties:

- particles from different populations can only produce particles from its own population;
- one-dimensional environments for different populations could be dependent.

For two-dimensional case we can represent $\{(Z_1(n), Z_2(n))\}_{n=0}^{\infty}$ in the following way:

$$(Z_1(n), Z_2(n)) = \sum_{i=1}^{Z_1(n-1)} (Y_1(i,n), 0) + \sum_{j=1}^{Z_2(n-1)} (0, Y_2(j,n)).$$

Critical case

We are interested in the case when both one-dimensional branching processes are critical.

Main example

Consider two one-dimensional i.i.d. environments $(Q_1(1), Q_1(2), \dots)$ and $(Q_2(1), Q_2(2), \dots)$ with corresponding generating functions $(F_1^{(1)}, F_1^{(2)}, \dots)$ and $(F_2^{(1)}, F_2^{(2)}, \dots)$:

$$F_1^{(i)}(s_1) = \sum_{k=1}^{\infty} \mathbf{P}(Q_1(i) = k)s_1^k = p_1(i) + \sum_{k=1}^{\infty}(1 - p_1(i))^k p_1(i)s_1^k$$

$$F_2^{(j)}(s_2) = \sum_{k=1}^{\infty} \mathbf{P}(Q_2(j) = k)s_2^k = p_2(j) + \sum_{k=1}^{\infty}(1 - p_2(j))^k p_2(j)s_2^k$$

Then $\mathbf{E}Q_1(1) = \frac{p_1}{1-p_1}$ and $\mathbf{E}Q_2(1) = \frac{p_2}{1-p_2}$.

$$X_1(i) := \log \mathbf{E}Q_1(i), \;\; X_2(i) := \log \mathbf{E}Q_2(i), \;\; S_1(n) := \sum_{k=1}^{n} X_1(k) \quad S_2(n) := \sum_{k=1}^{n} X_2(k),$$

where we set

$$\mathrm{Cor}\left[X_1(1)X_2(1)\right] = \varrho, \quad \varrho \in (1, 1).$$

We also consider $\mathbb{E}[X_1(1)^2] = \mathbb{E}[X_2(1)^2] = 1$.

Global goal

Survival probability and coexistence

We are considering which is called **coexistence** probability: $\mathbb{P}\big(Z_1(n) > 0, Z_2(n) > 0\big)$.

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Main example

For one-dimensional case we can represent $\mathbb{P}\big(Z(n) > 0)\big)$ in a more convenient form:

$$\mathbb{P}\big(Z(n) > 0)\big) = \left(\frac{1}{\mathbf{E}Q(1)\ldots\mathbf{E}Q(n)} + \sum_{k=1}^{n} \frac{\phi(Q(1),\ldots,Q(n))}{\mathbf{E}Q(1)\cdot\cdots\cdot\mathbf{E}Q(n)}\right)^{-1}$$

Recalling definition of $S(k)$ and using the expression above for the main example we can simplify:

$$\mathbb{P}\big(Z_1(n) > 0, Z_2(n) > 0\big) = \mathbb{E}\left[\Big(\sum_{k=1}^{n} e^{-S_1(k)}\Big)^{-1}\Big(\sum_{k=1}^{n} e^{-S_2(k)}\Big)^{-1}\right].$$

Condition to stay in cone

Main idea

One can see that

$$\mathbb{E}\left[\Big(\sum_{k=1}^{n} e^{-S_1(k)}\Big)^{-1}\Big(\sum_{k=1}^{n} e^{-S_2(k)}\Big)^{-1}\right]$$

$$\geqslant \mathbb{E}\left[\Big(\sum_{k=1}^{n} e^{-S_1(k)}\Big)^{-1}\Big(\sum_{k=1}^{n} e^{-S_2(k)}\Big)^{-1},\ \min_{k\leqslant n}\{S_1(k), S_2(k)\} > -R\right]$$

$$\times \mathbb{P}\Big(\min_{k\leqslant n}\{S_1(k), S_2(k)\} > -R\Big).$$

Stopping time

For positive quadrant $\mathcal{K}^2$ let τ_x be the first time n such that $x + S(n)$ leaves $\mathcal{K}^2$. Then we can rewrite the expression above as

$$\mathbb{E}\left[\Big(\sum_{k=1}^{n} e^{-S_1(k)}\Big)^{-1}\Big(\sum_{k=1}^{n} e^{-S_2(k)}\Big)^{-1},\ \tau_{(R,R)} > n\right] \cdot \mathbb{P}\big(\tau_{(R,R)} > n\big)$$

Random walks in cones

Main asymptotic

Let $\mathcal{K}^2$ be a strictly positive quadrant in $\mathbb{R}^2$. Let also $\mathrm{Cor}(X_1(1), X_2(1)) = \varrho$. Then:

$$\mathbb{P}(\tau_x > n) \sim V(x)\frac{1}{n^{\theta(\varrho)/2}},$$

where $\varrho \in (-1, 1)$ and $\theta(\varrho)$ is some function from ϱ.
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Harmonic property and upper bound

$V(x)$ is strictly positive in $\mathcal{K}^2$ and for arbitrary $x \in \mathcal{K}^2$ one has

$$\mathbb{E}\big(V(x+X), \quad x+X \in \mathcal{K}^2\big) - V(x) = 0.$$

Moreover, for arbitrary $x \in \mathcal{K}^2$ it is true that

$$V(x) \leqslant C \cdot |x|^{\theta(\varrho)-1} \cdot dist(x, \partial\mathcal{K}^2)$$

First result

Theorem 1

Assume that $|\varrho| < 1$. Set

$$\theta(\varrho) = \frac{\pi}{2\arccos(-\varrho)}.$$

Assume also that $\mathbb{E}|X(n)|^{2\theta}$ and $\mathbb{E}|X(n)|^2 \log(1 + |X(n)|)$ are finite. Then for every starting point z for branching process as in the main example there exists a positive constant $A = A(z)$ such that

$$\mathbb{P}_z(Z_1(n) > 0, Z_2(n) > 0) \sim An^{-\theta(\varrho)} \quad \text{as } n \to \infty. \tag{1}$$

Second result

Theorem 2

On the set $E(n) := \{Z_1(n) > 0, Z_2(n) > 0\}$ we define

$$z^{(n)}(t) = \left(\frac{\log Z_1(nt)}{\sqrt{n}}, \frac{\log Z_1(nt)}{\sqrt{n}} \right), \quad t \in [0,1].$$

Suppose all the conditions of Theorem 1 hold. Then the sequence $z^{(n)}$ conditioned on $E(n)$ converges in distribution on $D[0,1]$ with uniform metric. The limiting process can be called Brownian meander in $\mathcal{K}^2$.

Idea of the proof of the Theorem 1

Doob h-transform

For arbitrary $v \in \mathcal{K}^2$ one can consider Doob h-transform:

$$\hat{\mathbb{P}}(x + S(n) = v) := \frac{V(v)}{V(x)} \mathbb{P}(x + S(n) = v, \tau_x > n).$$

We want to estimate $\hat{\mathbb{P}}(x_1 + S_1(n) < h(n))$ for some function $h(\cdot)$.

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Estimation in crucial points

First, we are estimating small deviations of the form $\hat{\mathbb{P}}\big(x + S_2(n) < h(n), x + S_1(n) \leqslant \chi(n)\big)$. Main estimates are obtained from small enough values of $V(v)$.

Second, we are estimating large deviations of the form $\hat{\mathbb{P}}\big(x + S_2(n) < h(n), x + S_1(n) > \chi(n)\big)$. Main estimates are obtained from Fuk-Nagaev inequalities.

This estimates guarantee us that

$$\mathbb{E}\left[\Big(\sum_{k=1}^{n} e^{-S_1(k)}\Big)^{-1}\Big(\sum_{k=1}^{n} e^{-S_2(k)}\Big)^{-1}, \ \tau_{(R,R)} > n\right].$$