Interacting particle systems, conditioned random walks and the Aztec diamond

(based on arxiv:2310.18055)

Theo Assiotis

University of Edinburgh

Branching and Persistence conference Angers
April 9-11, 2025

PART I

Interacting particles, determinantal processes, conditioned walks

Introduction

- Model of interacting particles in interlacing array (Gelfand-Tsetlin pattern) in **space inhomogeneous** environment.
- Homogeneous model studied by Borodin-Ferrari and Warren-Windridge/Dieker-Warren.
- Particles move with continuous time jumps or discrete time Bernoulli or geometric jumps subject to interactions.
- Projections on edge particles (1+1)-dim inhomogeneous growth model, pushTASEP and TASEP-like systems.
- Projection on top row discrete Dyson Brownian motion.
- Particle and time (but not space!) inhomogeneities come up in integrable models of last passage percolation (LPP).
- Many connections. In part II will focus on Aztec diamond connection via the shuffling algorithm.

The basic data

- An inhomogeneity sequence $\mathbf{a} = (a_x)_{x \in \mathbb{Z}_+}$. Uniformly bounded in $(0, \infty)$.
- Particles live in $\mathbb{Z}_+$ and **a** inhomogeneity of space.
- To inhomogeneity **a** associate sequence of "**characteristic polynomials**" $(p_x)_{x \in \mathbb{Z}_+}$ given by

$$p_x(z) = \prod_{k=0}^{x-1} \left(1 - \frac{z}{a_k}\right).$$

- Inhomogeneities in time and of particles can also be introduced and keep the "integrability" of the model. But their role somewhat different.

The one-dimensional dynamics

- Three types of dynamics.
- **1. Cts time.** General pure-birth chain in continuous time. When at location x particle jumps to $x + 1$ with rate a_x.
- **2. Bernoulli.** Inhomogeneous Bernoulli jumps in discrete time. When at location x particle jumps to $x + 1$ with probability a_x and stays put with probability $1 - a_x$.
- Can add extra parameter α_t at time t. Jump probability to $x + 1$ at time t given by $\alpha_t a_x$ and stay at x with $1 - \alpha_t a_x$.
- **3. Geometric.** Inhomogeneous geometric jumps in discrete time. Particle at location x jumps to location $y \geq x$ with probability $(1 + a_y)^{-1} \prod_{k=x}^{y-1} a_k (1 + a_k)^{-1}$.
- Can think of passing each edge $(k, k + 1)$ as an independent trial with success probability $a_k \left(1 + a_k\right)^{-1}$.
- Can also add extra parameter β_t at time t.

Interlacing array

- Ordered configurations

$$\mathbb{W}_N = \{\mathbf{x} = (x_1, \dots, x_N) : 0 \le x_1 < x_2 < \cdots < x_N\}.$$

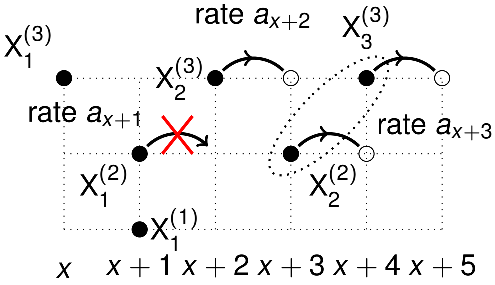
- $\mathbf{x} \in \mathbb{W}_N$ interlaces with $\mathbf{y} \in \mathbb{W}_{N+1}$, $\mathbf{x} \prec \mathbf{y}$, if

$$y_1 \le x_1 < y_2 \le x_2 < \cdots < y_N \le x_N < y_{N+1}.$$

- Interlacing array $\left(\mathbf{x}^{(i)}\right)_{i=1}^{M}$, M can be ∞, if $\mathbf{x}^{(i)} \prec \mathbf{x}^{(i+1)}$.
- Fully-packed configuration if $\mathbf{x}^{(n)} = (0, 1, \dots, n-1)$.
- Can think of all of the above in terms of partitions via coordinate shift.

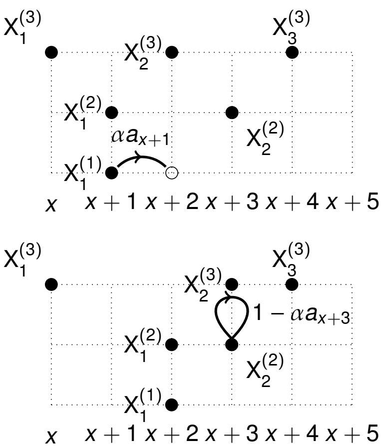
Dynamics in arrays I

- Each particle moves independently with above transition probabilities subject to certain *push-block interactions* to keep interlacing.
- In continuous time (no jumps at same time):

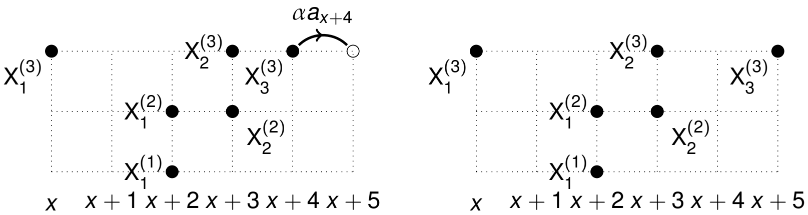


Dynamics in arrays II

- In discrete time we sequentially update each level, from bottom levels going up.
- For Bernoulli jumps:



Dynamics in arrays III



- Geometric step interactions similar but slightly more involved/subtle. Use particle location at previous time-step for blocking ("parallel update").
- In discrete time, each step can be either Bernoulli or geometric with an additional parameter α_t or β_t.

Autonomous edge particle systems

- The evolution of edge systems $\left(X_i^{(i)}\right)_{i=1}^N$ and $\left(X_1^{(i)}\right)_{i=1}^N$ is autonomous.
- Right edge system: inhomogeneous space and time pushTASEP with cts pure-jump, discrete-time Bernoulli or geometric jumps.
- Left edge system: inhomogeneous space and time zero-range/Boson particle system.
- Left edge under coordinate shift becomes a kind of inhomogeneous TASEP (but not inhomogeneous-space TASEP/slow bond problem!).
- Such particle systems arise from various integrable percolation models. **However**, the inhomogeneities of the environment become inhomogeneities of the particle (and time) instead of space (and time) as they are here!

A structure theorem

Theorem (A. 2023)

Consider $\left(\left(\mathsf{X}_i^{(n)}(t)\right)_{i,n}; t \geq 0\right)$ *following the cts-time dynamics or discrete-time dynamics with a mixture of time-inhomogeneous Bernoulli or geometric jumps starting from the fully-packed configuration. Then, we have:*

1. *The projection on any single level* $\left(\mathsf{X}^{(N)}; t \geq 0\right)$ *is Markovian with an explicit transition semigroup* $\mathfrak{P}_{s,t}^{(N)}$.
2. *Correlations of whole array at fixed time or of any single level at multiple times are determinantal, basically*

$$\mathbb{P}(\text{particles at locations } z_1, \ldots, z_m) = \det\left(\mathcal{K}(z_i, z_j)\right)_{i,j=1}^{m}.$$

   for explicit $\mathcal{K}$ *given as double-contour integral.*

- Everything works for family of "consistent" initial conditions.

A limit theorem: short-time asymptotics

- Using the above result can prove the following theorem:

Theorem (A. 2023)

Consider $\left(\left(X_i^{(n)}(t)\right)_{i,n}; t \geq 0\right)$ *following the cts time dynamics starting from fully-packed configuration. Assume the average* $\bar{a} = \lim_{N\to\infty} N^{-1}\sum_{x=0}^{N-1} a_x$ *exists. Let* $\zeta > 0$*. Then, for all* $m \geq 1$*,*

$$\left(X_{N-i+1}^{(N)}\left(\frac{\zeta}{N}\right) - N\right)_{i=1}^{m} \xrightarrow{\mathrm{d}} \left(\mathfrak{B}_i^{\zeta\bar{a}}\right)_{i=1}^{m}, \text{ as } N \to \infty,$$

where, for $\sigma > 0$, $\mathfrak{B}_1^\sigma > \mathfrak{B}_2^\sigma > \mathfrak{B}_3^\sigma > \cdots$ *are the ordered points of the discrete Bessel determinantal point process* $\mathfrak{B}^\sigma$ *on* $\mathbb{Z}$ *with correlation kernel* $\mathbf{J}_\sigma(x,y)$ *given by, with* $x, y \in \mathbb{Z}$,

$$\mathbf{J}_\sigma(x,y) = \frac{1}{(2\pi\mathrm{i})^2}\oint_{|z|=1^-}\oint_{|v|=1^+}\frac{z^x}{v^{y+1}}e^{z^{-1}-v^{-1}+\sigma v-\sigma z}\frac{1}{v-z}dzdv.$$

The semigroup $\mathfrak{P}_{s,t}^{(N)}$ I

- Explicit expression for the transition kernel

$$\mathfrak{P}_{s,t}^{(N)}(\mathbf{x},\mathbf{y}) = \frac{\mathfrak{h}_N(\mathbf{y})}{\mathfrak{h}_N(\mathbf{x})} \det\left(\mathfrak{P}_{s,t}^{(1)}(x_i, y_j)\right)_{i,j=1}^{N},$$

with $\mathfrak{h}_N(\mathbf{x}) = (-1)^{\frac{N(N-1)}{2}} \det\left(\partial_z^{i-1} p_{x_j}(z)\Big|_{z=0}\right)_{i,j=1}^{N}$.

- **Question** (work in progress). Does $\mathfrak{P}_{s,t}^{(N)}$ describe the evolution of N independent random walks, each with semigroup $\mathfrak{P}_{s,t}^{(1)}$, conditioned to never intersect?
- Doob h-transform structure does not immediately imply this specific conditioning. This conditioning gives rise to a new **Gibbs resampling property** (cf. Brownian Gibbs property) depending on **a**. Useful in taking N-limits.
- Think of as natural discrete (inhomogeneous) space analogue of **Dyson Brownian motion**.

The semigroup $\mathfrak{P}_{s,t}^{(N)}$ II

- Want to show: let τ_N be the first collision time. Then, for any fixed $0 \leq t < s$,

$$\lim_{s\to\infty} \frac{\mathbb{P}_{\mathbf{y}}\left(\tau_N > s - t\right)}{\mathbb{P}_{\mathbf{x}}\left(\tau_N > s\right)} \stackrel{?}{=} \frac{\mathfrak{h}_N\left(\mathbf{y}\right)}{\mathfrak{h}_N\left(\mathbf{x}\right)}.$$

- Note that individual probabilities $\mathbb{P}_{\mathbf{x}}\left(\tau_N > s\right)$ converge to 0 as $s \to \infty$. Subtle cancellation in taking the ratio.
- I have already proven a modification of this statement, when adding different ordered “drifts” to each walk.
- Makes the asymptotics problem easier as non-collision probability converges to strictly positive explicit limit. With prob 1 walks become asymptotically ordered based on the order of their drifts.
- Taking the limit of all “drifts” to zero formally recovers the desired statement above.

On the proof of structure theorem I

- First prove that the semigroups $\mathfrak{P}^{(N+1)}_{s,t}$ and $\mathfrak{P}^{(N)}_{s,t}$ are intertwined:

$$\mathfrak{P}^{(N+1)}_{s,t}\mathfrak{L}^{\mathbf{a}}_{N+1,N} = \mathfrak{L}^{\mathbf{a}}_{N+1,N}\mathfrak{P}^{(N)}_{s,t}.$$

- The Markov kernels (not obvious that they are indeed Markov) $\mathfrak{L}^{\mathbf{a}}_{N+1,N}$ from $\mathbb{W}_{N+1}$ to $\mathbb{W}_N$ are given by,

$$\mathfrak{L}^{\mathbf{a}}_{N+1,N}(\mathbf{y},\mathbf{x}) = \frac{\mathfrak{h}_N(\mathbf{x})}{\mathfrak{h}_{N+1}(\mathbf{y})}\prod_{i=1}^{N}\frac{1}{a_{x_i}}\mathbf{1}_{\mathbf{x}\prec\mathbf{y}}.$$

- Given two intertwined semigroups there are many ways to couple them. General recipes exist e.g. Diaconis-Fill construction. Later developed by Borodin, Ferrari, Corwin, Petrov, ... Some couplings less natural than others e.g. projections on edge systems may not be Markovian.

On the proof of structure theorem II

- Coupling here constructed from formula for transition probabilities of bi-variate chain “arising” from coalescing random walks in a uniform way for all three types of motion.
- Matches the coupling of Diaconis-Fill for cts time and Bernoulli jumps. But **different for geometric jumps**! D-F geometric coupling gives non-Markovian left edge system.
- In homogeneous case $a_x \equiv 1$ the “coalescing walk coupling” matches coupling of Warren-Windridge.
- Finally to compute correlation kernel need to solve a certain bi-orthogonalisation problem.

Main computational tool

- Generalisation of a Toeplitz matrix $[\mathrm{T}_f(x,y)]_{x,y\in\mathbb{Z}_+}$ given by:

$$\mathrm{T}_f(x,y) = -\frac{1}{2\pi\mathrm{i}}\frac{1}{a_y}\oint_{\mathrm{C}_{\mathbf{a}}}\frac{p_x(z)}{p_{y+1}(z)}f(z)dz,$$

  with $\mathrm{C}_{\mathbf{a}}$ counter-clockwise contour encircling $\{a_x\}_{x\in\mathbb{Z}_+}$.
- For $a_x \equiv 1$ this is just Toeplitz matrix with symbol $f(1-z)$.
- Similar to a Toeplitz matrix (i.e. $a_x \equiv 1$) with explicit but complicated change of basis matrix $\mathbf{A}(\mathbf{a})$. Similarity not very useful for our purposes though.
- Probabilistic applications given by (cf. 1-d semigroup $\mathfrak{P}^{(1)}_{s,t}$):

$$f(z) = \begin{cases} e^{-tz}, & \text{pure-birth chain, cts time,} \\ 1-\alpha z, & \text{Bernoulli walk, discrete time,} \\ (1+\beta z)^{-1}, & \text{geometric walk, discrete time.} \end{cases}$$

- Could also consider matrix symbol $\mathbf{f}$.

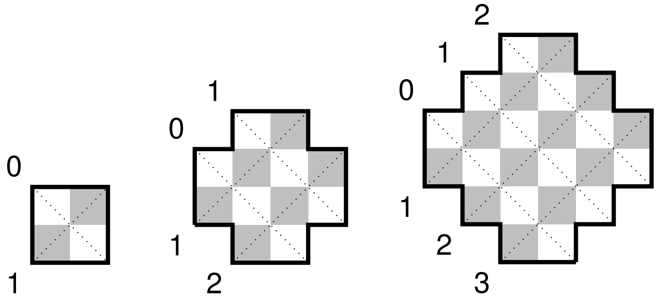
PART II

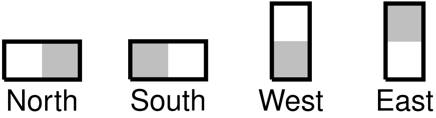
Bernoulli dynamics and the Aztec diamond

The Aztec diamond I

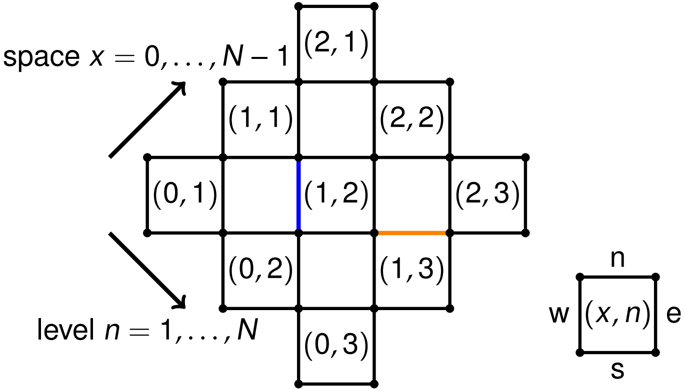
- Combinatorial model introduced by Elkies, Kuperberg, Larsen, Propp in ’92. Enormous amount of work since.
- Sawtooth domain in $\mathbb{Z}^2$.
- Can tile area with horizontal and vertical dominoes.
- Equivalent to dimer cover of Aztec diamond graph.
- Aztec diamond graph of size N consists of N^2 squares.
- We will put a coordinate system on the graph.

The Aztec diamond II



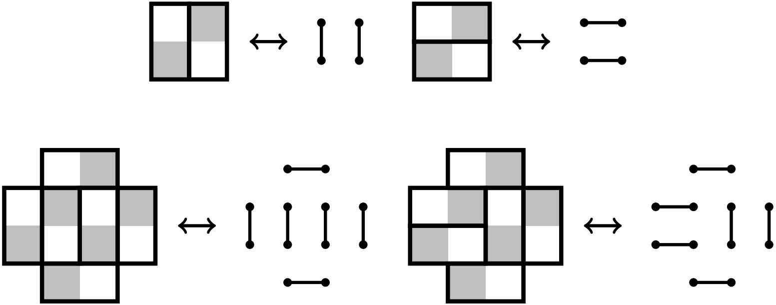


The Aztec diamond III



- The Aztec diamond graph of size $N = 3$, along with the corresponding coordinate system.
- Coordinates of the blue vertical edge are $(\mathrm{w}, (1,3))$, while coordinates of the horizontal orange edge are $(\mathrm{n}, (1,3))$.

The Aztec diamond IV



The Aztec diamond V

- A weighting $\mathcal{W}$ of Aztec diamond graph of size N is a function from its edge set to $(0, \infty)$.
- Write $\mathcal{W}_e$ for its value at the edge e.
- Given a weighting $\mathcal{W}$ of Aztec diamond graph of size N define the probability measure $\mathbb{P}_{\mathcal{W}}^{(N)}$ on dimer coverings of the graph of size N by:

$$\mathbb{P}_{\mathcal{W}}^{(N)}(\text{dimer cover}) = \frac{1}{Z_{\mathcal{W}}} \prod_{e \in \text{dimer cover}} \mathcal{W}_e,$$

where $Z_{\mathcal{W}} = \sum_{\text{all dimer covers}} \prod_{e \in \text{dimer cover}} \mathcal{W}_e$ is the normalisation constant/partition function.

The Aztec diamond VI

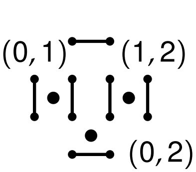
- Given such $\mathcal{W}$ associate probabilities $\rho_{\mathcal{W}}$ to each square (x, n) by:

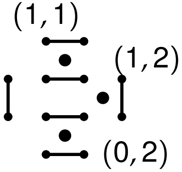
$$\rho_{\mathcal{W}}(x,n) = \frac{\mathcal{W}_{\mathrm{w},(x,n)}\mathcal{W}_{\mathrm{e},(x,n)}}{\mathcal{W}_{\mathrm{w},(x,n)}\mathcal{W}_{\mathrm{e},(x,n)} + \mathcal{W}_{\mathrm{n},(x,n)}\mathcal{W}_{\mathrm{s},(x,n)}}.$$

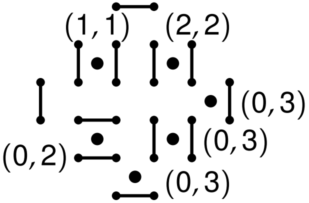
- Gauge-equivalence: Multiplying all edges incident to a vertex by a number in $(0, \infty)$ does not change the probability measure on tilings or the square probabilities.
- From probabilistic standpoint gauge-equivalent weightings are the same.

Particle system from tiling

- Given a dimer cover, associate a particle configuration by putting a particle in each square with a south or east edge.
- The particle inherits the coordinates of the square.
- Fact that there are exactly n particles at level n.







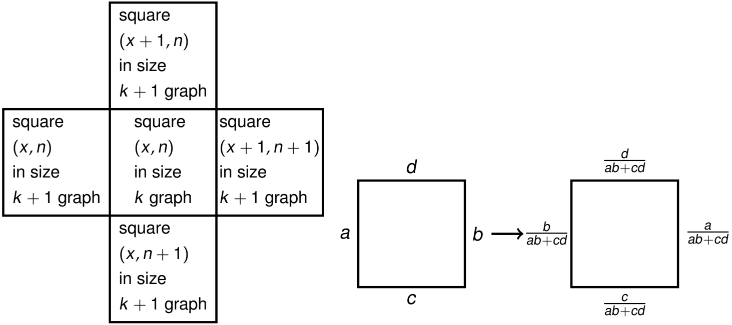
Urban renewal/spider move I

- Given $\mathcal{W}$ weighting of size k Aztec diamond graph.
- Urban renewal/spider move map $\mathcal{UR}_{k-1}^{k}$ constructs weighting $\mathcal{UR}_{k-1}^{k}(\mathcal{W})$ of size $k-1$ graph.
- More generally, define the maps for $m < k$,

$$\mathcal{UR}_{m}^{k} = \mathcal{UR}_{m}^{m+1} \circ \cdots \circ \mathcal{UR}_{k-1}^{k}.$$

Urban renewal/spider move II

- Illustration of the urban renewal map for a single square.



Some motivation

- The fact that random tilings of the Aztec diamond with general weighting $\mathcal{W}$ have determinantal correlations is well-known from Kasteleyn theory.
- A lot of effort to compute kernel explicitly, by variety of methods, in form that can be analysed asymptotically (Kenyon, Johansson, Borodin, Ferrari, Chhita, Duits, Kuijlaars, Berggren,...).
- Central example two-periodic weighting. Gives rise to gas/smooth (exponential decay of correlations) region in the limit.
- Some motivation behind what comes next: try to find some **conditioned random walk/Gibbs resampling property structure** behind random tilings of Aztec diamond with weighting $\mathcal{W}$.
- To begin with, we need some dynamics.

The shuffling algorithm

- This is a perfect sampling algorithm of a random $\mathbb{P}^{(N)}_{\mathcal{W}}$-distributed tiling of the size N Aztec diamond.
- Start from size 0 Aztec diamond. Sequentially, from tiling of Aztec diamond of size k, we create a tiling of Aztec diamond of size $k+1$.
- Each iteration of the algorithm consists of 4 steps.
  1. Embedding.
  2. Deletion.
  3. Sliding.
  4. Creation.
- This induces random evolution of corresponding particles.
- Steps 1-3 are deterministic.
- *Probability and dependence on $\mathcal{W}$ only comes in Step 4*.
- Theorem (Propp) that when algorithm terminates, the resulting size N Aztec diamond tiling is $\mathbb{P}^{(N)}_{\mathcal{W}}$-distributed.

Embedding

- Embed graph of size k into graph of size $k+1$ so that square (x, n) in graph of size k consists of the west, north, east and south edges of squares (x, n), $(x, n+1)$, $(x+1, n+1)$ and $(x+1, n)$ in size $k+1$ graph respectively.
- This embeds dimer covering of graph of size k into a subcollection of edges of graph of size $k+1$.

Deletion

- If in this embedding two dimers of the dimer covering belong to the same square of the size $k+1$ graph (in this embedding) we remove them.

Sliding

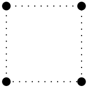
- Then move all dimers by one edge in the opposite direction of their names, viewed as dimers in the size $k+1$ graph.
- Namely, a north dimer moves down by one, a south dimer moves up by one, a west dimer moves right by one and an east dimer moves left by one.

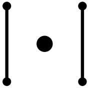
Creation

- This leaves a number of squares not covered by any dimers which are filled in the following fashion.
- If square (x, n) is empty it is covered with a west-east dimer pair with probability $\rho_{\mathcal{UR}^N_{k+1}(\mathcal{W})}(x, n)$ and covered with a north-south dimer pair with probability $1 - \rho_{\mathcal{UR}^N_{k+1}(\mathcal{W})}(x, n)$.
- This gives a dimer covering of the size $k + 1$ graph. In fact, this dimer covering is $\mathbb{P}^{(k+1)}_{\mathcal{UR}^N_{k+1}(\mathcal{W})}$-distributed.

Shuffle in action: Iteration 1

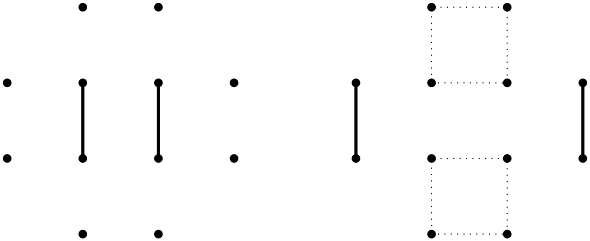
- Suppose we have $\mathcal{W}$ weighting of Aztec diamond of size 3.
- We want to sample tiling according to $\mathbb{P}^{(3)}_{\mathcal{W}}$.
- Run urban renewal to create $\mathcal{UR}_2^3(\mathcal{W})$ and $\mathcal{UR}_1^3(\mathcal{W})$. Recall we will need the corresponding square probabilities.

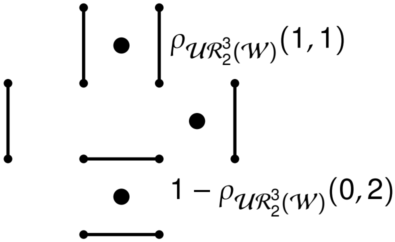




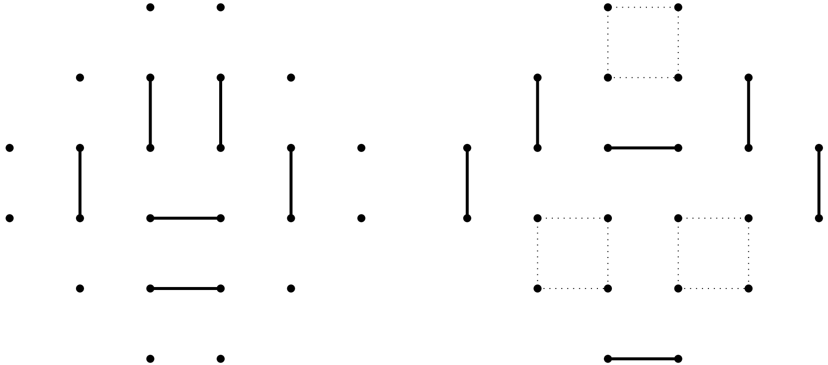
$\rho_{\mathcal{UR}_1^3(\mathcal{W})}(0,1)$

Shuffle in action: Iteration 2

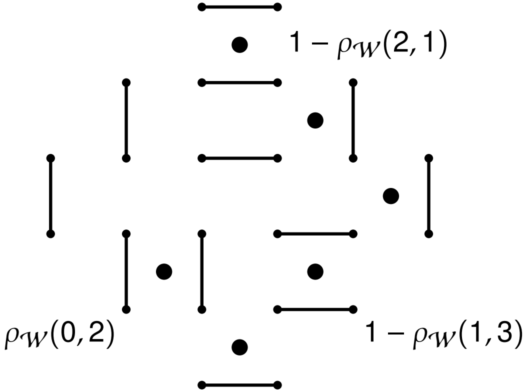




Shuffle in action: Iteration 3 part I



Shuffle in action: Iteration 3 part II



Dynamics on arrays from shuffle I

- Let $\mathcal{W}$ be a weighting of size N Aztec diamond.
- Define for $j \leq t \leq N$, $1 \leq i \leq j$, $\mathrm{x}_i^{(j),\mathrm{sh}}(t)$ to be position of i-th particle of level j after t steps of the shuffle.
- On the other hand, consider the push-block Bernoulli dynamics **except** that a particle at space location x, at level n, at time t has jump probability $\rho_{\mathcal{UR}^N_{t+n}(\mathcal{W})}(x, n)$ instead.
- This defines a stochastic process

$$\left(\mathrm{Y}_i^{(j)}(t); 0 \leq t \leq N - j, 1 \leq i \leq j, 1 \leq j \leq N\right).$$

Dynamics on arrays from shuffle II

Proposition

Let $N \geq 1$. Let $\mathcal{W}$ be a weighting of size N Aztec diamond. Let $\mathsf{Y}_i^{(j)}(t)$ and $\mathsf{x}_i^{(j),\mathrm{sh}}(t)$ be as above. Then, we have the following equality in distribution, jointly in all involved indices,

$$\left(\mathsf{Y}_i^{(j)}(t-j); 1 \leq j \leq N, 1 \leq i \leq j, j \leq t \leq N\right) \stackrel{\mathrm{d}}{=} \left(\mathsf{x}_i^{(j),\mathrm{sh}}(t); 1 \leq j \leq N, 1 \leq i \leq j, j \leq t \leq N\right).$$

- Uniform weight case due to Nordenstam.
- Uniform case restricted to edge particle systems: original work of Jokusch, Propp, Shor.

Coupling Aztec diamonds of all sizes

- Want to couple random tilings of Aztec diamonds of all sizes. This gives particle system on infinite arrays living for all times.
- Need the notion of **sequence of consistent weightings**: $\left(\mathcal{W}^{(m)}\right)_{m=1}^{\infty}$ such that for all $m \geq 1$,

$$\mathcal{W}^{(m)} \text{ is gauge-equivalent to } \mathcal{UR}_m^{m+1}\left(\mathcal{W}^{(m+1)}\right).$$

- Basically this means that random tiling obtained by running m steps of the shuffle is $\mathbb{P}^{(m)}_{\mathcal{W}^{(m)}}$-distributed.
- Easy to see that sequence of uniform weights is consistent. In general need some computation to check.

A special inhomogeneous weight

- Suppose $\mathbf{z}^{(1)}, \mathbf{z}^{(2)} \in (0, \infty)^{\mathbb{Z}_+}$ such that

$$\frac{z_x^{(1)}}{z_x^{(1)} + z_x^{(2)}} = a_x.$$

- Consider, for any $k \geq 1$, the following weighting $\mathcal{W}^{(k),\mathbf{a}}$,

$$\mathcal{W}^{(k),\mathbf{a}}_{\mathrm{e},(x,n)} = \mathcal{W}^{(k),\mathbf{a}}_{\mathrm{n},(x,n)} = 1, \mathcal{W}^{(k),\mathbf{a}}_{\mathrm{w},(x,n)} = z_x^{(1)}, \mathcal{W}^{(k),\mathbf{a}}_{\mathrm{s},(x,n)} = z_x^{(2)}.$$

- Inhomogeneous **only** in the space direction.
- Not hard to show that the sequence $\left(\mathcal{W}^{(k),\mathbf{a}}\right)_{k=1}^{\infty}$ is consistent. Moreover the square probabilities $\rho_{\mathcal{W}^{(k),\mathbf{a}}}$ are

$$\rho_{\mathcal{W}^{(k),\mathbf{a}}}(x, n) = a_x.$$

- Inhomogeneous model in other direction (level) gives particle-dependent model of B-F. Aztec diamond shuffle gives some sort of duality between the two (time-shifted) dynamics.

Conditioned walks and the Aztec diamond

Theorem (A. 2023)

Consider the probability measures $\mathbb{P}^{(k)}_{\mathcal{W}^{(k),\mathbf{a}}}$ associated to $\mathcal{W}^{(k),\mathbf{a}}$. Then, there exists a coupling (shuffle dynamics!) of the $\mathbb{P}^{(k)}_{\mathcal{W}^{(k),\mathbf{a}}}$ such that if $\mathsf{x}_i^{(j)}(m)$, for $m \geq j$, is the location of the i-th south or east dimer on level j of the tiling distributed according to $\mathbb{P}^{(m)}_{\mathcal{W}^{(m),\mathbf{a}}}$ in this coupling, then jointly for all $N \geq 1$,

$$\left(\mathsf{x}_1^{(N)}(t+N), \mathsf{x}_2^{(N)}(t+N), \dots, \mathsf{x}_N^{(N)}(t+N); t \geq 0\right)$$

evolves as a Markov process in $\mathbb{W}_N$ with explicit transition probabilities. Moreover, the correlations of this process in time are determinantal with an explicit correlation kernel.

- Proof of probabilistic statement rather roundabout by connecting to dynamics on arrays. Is there a direct proof?
- Computation of kernel can also be obtained by other methods alluded to earlier.

Some questions

- Scaling limits: other local limits or global limits?
- Infinite line ensemble limit of inhomogeneous walks conditioned to never intersect?
- Conditioned random walks structure in two-periodic Aztec diamond?

Thank You

Thank you for your attention