Shape of a leaky sandpile model via a killed random walk

Théo Ballu, joint work with Cédric Boutillier, Sevak Mkrtchyan and Kilian Raschel

LAREMA, Université d'Angers

Branching and Persistence, Angers, April 2025

Summary

The leaky sandpile model on $\mathbb{Z}^d$

The limit shape as the amount of sand goes to $+\infty$

Influence of the parameter ε

A bit of history

The sandpile model is a cellular automaton coming from theoretical physics.

- 1989 : notion of *self-organized criticality* in [BakTangWies][1].
- 1990 : sandpile model in [Dhar][2]

[1] Per Bak, Chao Tang **and** Kurt Wiesenfeld. "Self-organized criticality: An explanation of the 1/f noise". **inPhys. Rev. Lett.**: ().

[2] Deepak Dhar. "Self-organized critical state of sandpile automaton models". **inPhys. Rev. Lett.**: ().

The leaky abelian sandpile model

Let $G = \mathbb{Z}^d$ with periodic edges.

- An amount N of sand is placed at the origin.

The leaky abelian sandpile model

Let $G = \mathbb{Z}^d$ with periodic edges.

- An amount N of sand is placed at the origin.
- Let $x \in \mathbb{Z}^d$. If the amount of sand exceeds a threshold, then x is **unstable** and **topples**. It means that :

The leaky abelian sandpile model

Let $G = \mathbb{Z}^d$ with periodic edges.

- An amount N of sand is placed at the origin.
- Let $x \in \mathbb{Z}^d$. If the amount of sand exceeds a threshold, then x is **unstable** and **topples**. It means that :
  - it sends an amount $c(y)$ grains of sand to the vertex $x + y$ for every $y \in \mathbb{Z}^d$ (where $c(y) \neq 0$ iff xy is an edge),
  - a proportion $\varepsilon \in (0, 1)$ of the sand is lost.

The leaky abelian sandpile model

Let $G = \mathbb{Z}^d$ with periodic edges.

- An amount N of sand is placed at the origin.
- Let $x \in \mathbb{Z}^d$. If the amount of sand exceeds a threshold, then x is **unstable** and **topples**. It means that :
  - it sends an amount $c(y)$ grains of sand to the vertex $x + y$ for every $y \in \mathbb{Z}^d$ (where $c(y) \neq 0$ iff xy is an edge),
  - a proportion $\varepsilon \in (0, 1)$ of the sand is lost.
- As long as there are unstable vertices, they topple until the configuration becomes stable.

Remarks

- The sand spreads from the center outwards.
- When several vertices are unstable at the same time, the order of topplings does not modify the final stable configuration.
- The toppling process is **deterministic**.

One toppling in $\mathbb{Z}^2$

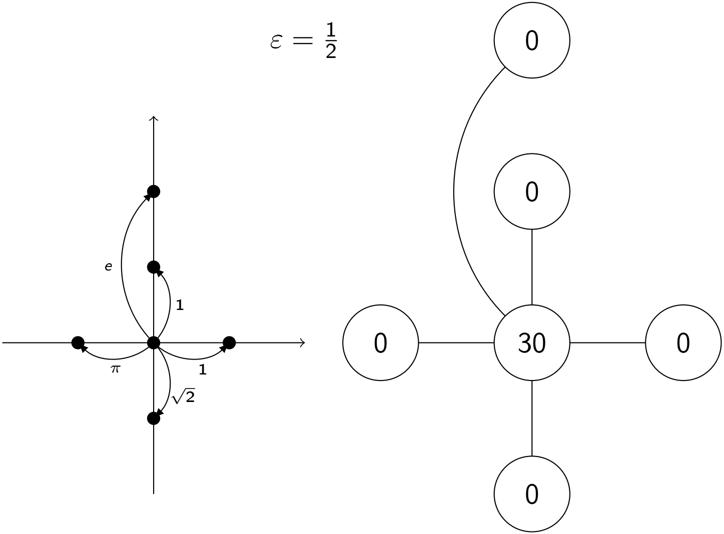
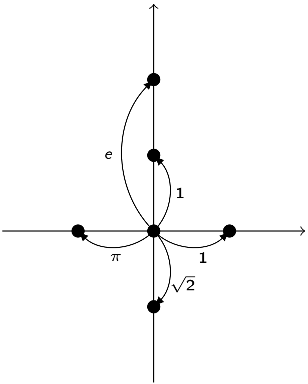


Figure: Left: amount of sand sent to neighbors at each toppling. Right: piece of an initial configuration in $\mathbb{Z}^2$.

One toppling in $\mathbb{Z}^2$

$\varepsilon = \frac{1}{2}$



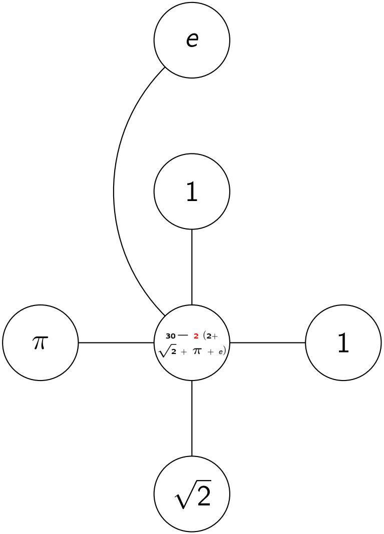


Figure: Configuration after the toppling of the vertex with 30 grains of sand.

The leaky sandpile model on $\mathbb{Z}^d$

The limit shape as the amount of sand goes to $+\infty$

Influence of the parameter ε

Simulations in 3D

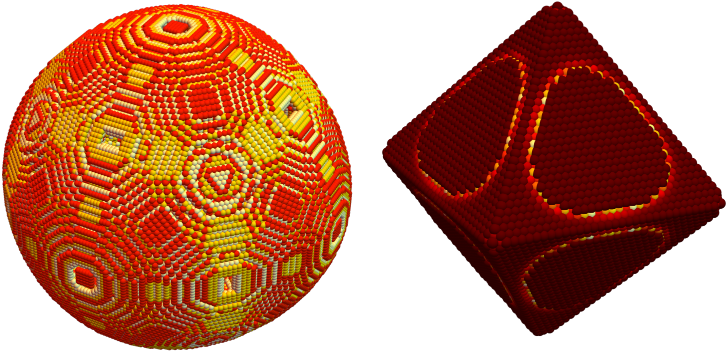


Figure: Simulations in 3D with different parameters ε.

An animation

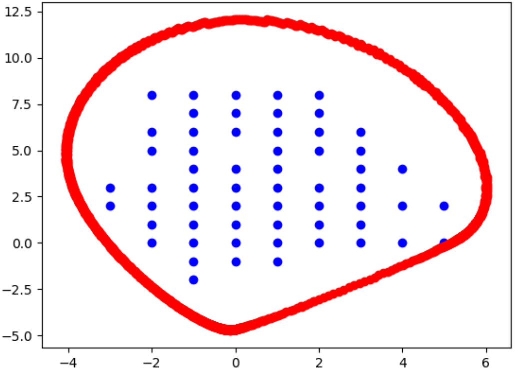


Figure: The original amount of sand goes from 10^3 to 10^{11}.

Killed random walk associated with the model

A **killed** random walk $(X_n)_{n\geq 0}$ on $\mathbb{Z}^d$ is associated with the model.

- At each step, it is killed with probability ε.
- Otherwise, it jumps from x to y with a probability proportional to the amount of sand sent by x to y.

Killed random walk associated with the model

A **killed** random walk $(X_n)_{n\geq 0}$ on $\mathbb{Z}^d$ is associated with the model.

- At each step, it is killed with probability ε.
- Otherwise, it jumps from x to y with a probability proportional to the amount of sand sent by x to y.

Its **Green function** is

$$G(0,x) = \mathbb{E}_0\left[\sum_{n=0}^{\infty} \mathbb{1}_{X_n=x}\right].$$

Thresholds for the Green function

Proposition

There exist two constants $\alpha, \beta > 0$ such that:

- if $G(0, x) > \frac{\alpha}{N}$, then x is in the final configuration of the sandpile;
- if $G(0, x) < \frac{\beta}{N}$, then x is **not** in the final configuration of the sandpile.

Illustration of the thresholds

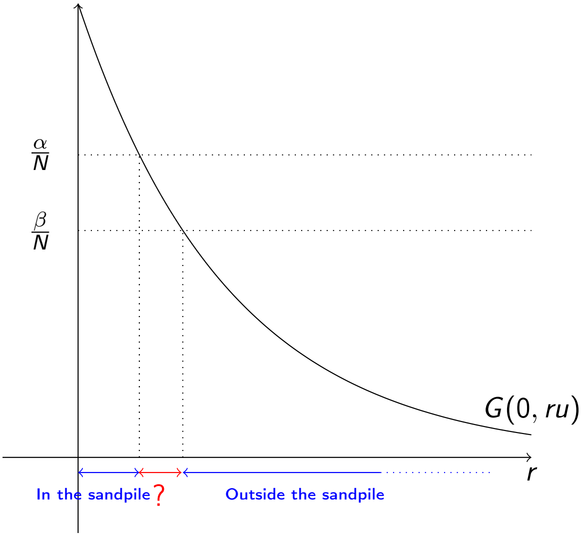
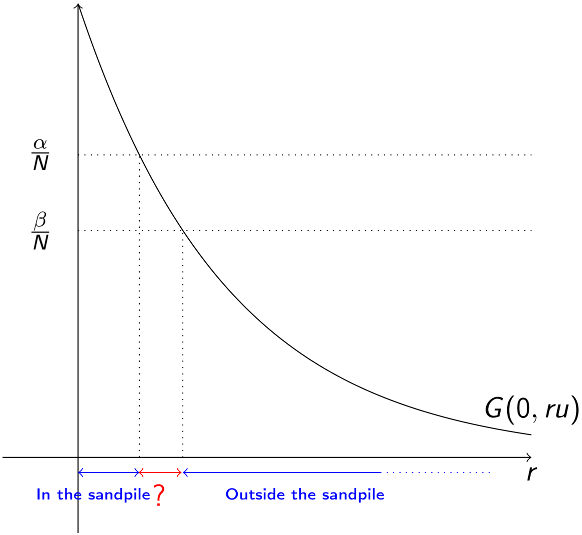


Illustration of the thresholds



If G decreases fast enough, the "?" zone will be small. Good news is that the Green function has exponential decay.

Convergence of the shape

Theorem

The sandpile has a limit shape as $N \to +\infty$ (after normalization by $\log(N)$).

Convergence of the shape

Theorem
The sandpile has a limit shape as $N \to +\infty$ (after normalization by $\log(N)$).

The limit shape can be expressed with the exponential decay of the Green function, or as the dual curve of level lines of the Laplace transform of the walk.

The leaky sandpile model on $\mathbb{Z}^d$

The limit shape as the amount of sand goes to $+\infty$

Influence of the parameter ε

Influence of the parameter ε

We first let $N \to +\infty$, and look at the cases $\varepsilon \to 1$ and $\varepsilon \to 0$.

Theorem

When $\varepsilon \to 1$, the limit shape converges to a polytope.
When $\varepsilon \to 0$, the limit shape converges to an ellipsoid.

Animations

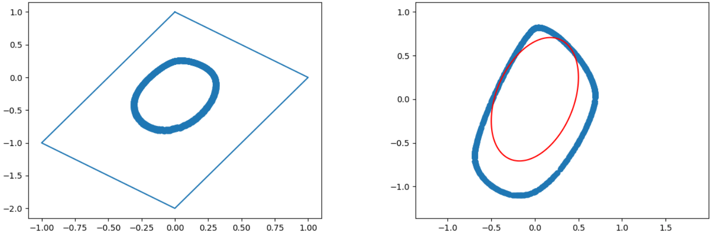


Figure: Left: $\varepsilon \to 0$. Right: $\varepsilon \to 0$.